UNIVERSITY OF LONDON

GOLDSMITHS COLLEGE

Department of Computing

B. Sc. Examination 2020

IS51032A
Symbolic Mathematics

Duration: 2 hours 15 minutes

Date and time:

This paper is in two parts: part A and part B. You should answer ALL questions from part A and TWO questions from part B. Part A carries 40 marks, and each question from part B carries 30 marks. The marks for each part of a question are indicated at the end of the part in [.] brackets.

There are 100 marks available on this paper.

The use of calculators is allowed. Students are required to note the model of the calculators on the answer sheet.

Electronic calculators must not be programmed prior to the examination. Calculators which display graphics, text or algebraic equations are not allowed.

THIS PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Part A

Multiple choice

Question 1 Each question has one or more correct answers

(a) Let $A = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}\}$. Which of the following sets represent A using the inclusion rules? More than one answer may apply.

i. $\{\frac{1}{2n} : n \in \mathbb{Z} \textbf{ and } 1 \leq n \leq 8\}$

ii. $\{2n^{-1} : n \in \mathbb{Z} \textbf{ and } 0 < n < 9\}$

iii. $\{\frac{1}{2^n} : n \in \mathbb{Z} \textbf{ and } 0 < n \leq 8\}$

iv. $\{\frac{1}{2^n} : n \in \mathbb{Z} \textbf{ and } 0 \leq n < 8\}$

[2]

(b) Let $S = \{a, b, c\}$, which one of the following sets represents $\mathcal{P}(S)$?

i. $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

ii. $\{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

iii. $\{\{a\}, \{b\}, \{c\}, \{a, b\}, a, b, \{b, c\}\}$

iv. $\{\{a\}, \{b\}, \{c\}\}$

[2]

(c) Let A be a set with n elements. What is the number of subsets which can be formed from A?

Choose ONE option. [2]

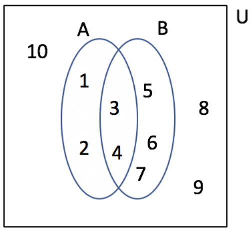
i. 2^n

ii. $2n$

iii. n^2

iv. $\frac{n}{2}$

(d) Given the following Venn diagram representing two sets A and B, subsets of the universal set U:



Which one of the following sets represents $(\overline{A \oplus B})$?

Choose ONE option. [4]

i. $\{1, 2, 5, 6, 7\}$

ii. $\{3, 4, 8, 9, 10\}$

iii. $\{8, 9, 10\}$

iv. none of the other options is correct

(e) Given the statement $S(x)$: '2 divides $x^2 + 1$', select the right statement from the following.
Choose ONE option. [2]

i. S can be expressed using propositional logic

ii. S is not a proposition, as its truth value is a function depending on x

iii. The truth value of $S(3)$ is False

iv. The truth value of $S(2)$ is True

(f) Let p and q be two propositions where p means '**Sofia is Happy**' and q means '**Sofia is reading a book**'. Which one of the following logical expressions is a correct formalisation of the following sentence:

Sofia is either happy or is reading a book, but not both

i. $\neg p \to q$

ii. $q \to \neg p$

iii. $\neg p \oplus q$

iv. $p \oplus q$

[2]

(g) Let p and q be the following propositions where p means 'James has a cell phone' and q means 'James has a laptop computer' Which one of the following logical expressions is equivalent to a correct formalisation of the sentence below?

'James does not have a cellphone or he does not have a laptop computer.'

Choose ONE option. [2]

i. $\neg(p \vee q)$

ii. $\neg p \wedge \neg q$

iii. $\neg(p \wedge q)$

iv. $\neg p \vee q$

(h) Let p and q be two propositions. Which one of the following compound statements is equivalent to $p \vee \neg(p \wedge q)$

i. $(p \vee \neg p) \wedge (p \vee \neg q)$

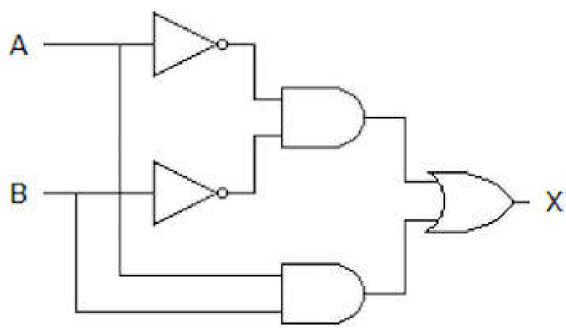
ii. $\neg p \vee \neg q$

iii. T

iv. F

[2]

(i) Which one is a correct output of the following logic network:



i. $(A \wedge B) \vee (\neg A \wedge \neg B)$

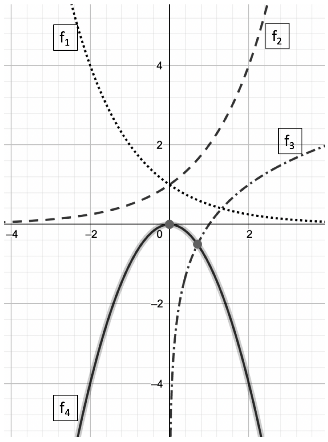
ii. $(A \wedge B) \vee (\neg A \wedge B)$

iii. $(A \wedge B) \vee (\ A \wedge \vee B)$

iv. $(A \vee B) \wedge (\neg A \vee \neg B)$

[4]

(j) The following graph shows the curves of four functions, f_1, f_2, f_3 and f_4:



Which one of these functions is not invertible?

Choose ONE option. [2]

i. f_1
ii. f_2
iii. f_3
iv. f_4

(k) Let $f : R^+ \to R$ be a function where $f(x) = \log_2 x$. Which one of the following is the inverse function of the function f?

i. $f^{-1}(x) = e^x$
ii. $f^{-1}(x) = 2^x$
iii. $f^{-1}(x) = \sqrt{x}$
iv. $f^{-1}(x) = \frac{x}{2}$

[2]

(l) The following sequence $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \cdots$ is

i. arithmetic
ii. geometric
iii. neither geometric nor arithmetic
iv. both arithmetic and geometric

[2]

(m) Which one of the following degree sequences cannot represent a simple graph? . [2]

i. $3, 3, 2, 2$

ii. $2, 2, 2, 2$

iii. $1, 1, 1, 1$

iv. $4, 3, 3, 2$

(n) Which of the following statements is/are **TRUE**? More than one answer might apply.

i. it is possible to draw a 5-regular graph with 5 vertices

ii. it is possible to draw 2-regular graph with 5 vertices

iii. the sum of the degree sequence of a graph is twice the number of edges in the graph

iv. the sum of the degree sequence of a graph is twice the number of vertices in the graph.

[2]

(o) The number of of edges in a complete graph with n vertices is

i. n-1

ii. n(n-1)

iii. n(n-1)/2

iv. 2n

[2]

(p) Which one the following correctly defines a Hamiltonian path? [2]

i. A Hamiltonian path in a graph G is a path that uses each edge in G precisely once

ii. A Hamiltonian path in graph G is a path that visits each vertex in G exactly once

iii. A Hamiltonian path path is a trail in which neither vertices nor edges are repeated

iv. A Hamiltonian path is a walk in which no edge is repeated

(q) Which one the following correctly defines an Eulerian path ? [2]

i. An Eulerian path in a graph G is a path that uses each edge in G precisely once

ii. $2, 2, 2, 2$

iii. An Eulerian path in graph G is a path that visits each vertex exactly once.

iv. An Eulerian path is a trail in which neither vertices nor edges are repeated

v. An Eulerian path is a walk in which no edge is repeated

(r) Let $S = \{1, 2, 3\}$ and $\mathcal{R}$ be a relation on elements in S with

$$\mathcal{R} = \{(1,1), (1,2), (2,1), (2.2), (3,3)\}$$

Which one of the following statements is correct about the relation $\mathcal{R}$?

[2]

i. $\mathcal{R}$ is reflexive and symmetric

ii. $\mathcal{R}$ is reflexive and anti-symmetric

iii. $\mathcal{R}$ is anti-symmetric and transitive

iv. $\mathcal{R}$ is reflexive symmetric and anti-symmetric

Part B

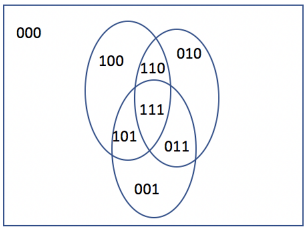
Question 2 Set, Logic & Sequences

(a) i. Rewrite the following three sets using the listing method:

- $A = \{n^2 - 1 : n \in \mathbb{Z} \textbf{ and } 0 \leq n < 4\}$
- $C = \{1 + (-1)^n : n \in \mathbb{Z}^+\}$

[4]

ii. Given the following Venn diagram representing three sets A, B and C intersecting in the most general way. Three binary digits are used to refer to each one of the 8 regions in this diagram. In terms of A, B and C, Write the set representing the area comprising the regions 100, 010 and 001. The answer must be written in its simplest form using the $\oplus$ operator only.



[4]

iii. Given two sets, A and B. Show that $A \cap \overline{(A \cup B)} = \emptyset$.

[2]

(b) i. Let p and q be two propositions for which $p \to q$ is false. Determine the truth value of for each of the following: $p \wedge q$; $\neg p \vee q$; $q \to p$ and $\neg q \to \neg p$.

[4]

ii. Let p, q and r denote the following statements:

p : 'I finish writing my computer program before lunch'

q : 'I play football in the afternoon'

r : 'The sun is shining'

1. Write the following the following statements into their corresponding symbolic forms.

- If the sun is shining and I finish writing my computer program, I shall play football this afternoon.
- I will play football this afternoon only if the sun is shining and I finish writing my computer program.

[4]

2. Write in words the contrapositive of the the following statement: 'If the sun is shining and I finish writing my computer program, I shall play football this afternoon.' [2]

(c) i. Consider the Fibonacci recurrence relation:

$$f_n = f_{n-1} + f_{n-2} \text{ with } f_0 = 0 \text{ and } f_1 = 1$$

What are the values of f_2 and f_3? [2]

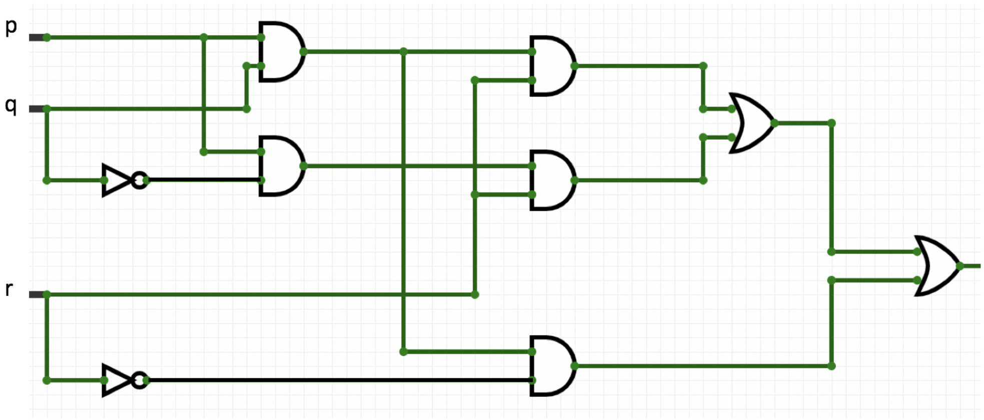
ii. Let $S_n = \sum_{i=1}^{i=n} (2i-1) = n^2$ for all $n \in \mathbb{Z}^+$.

1. Find S_1 and S_2. [2]

2. Prove by induction that $S_n = n^2$ for all $n \in \mathbb{Z}^+$. [6]

Question 3 Boolean Algebra & Functions

(a) Given the following logical circuit:



i. Identify the logical gates used in this circuit. [2]

ii. What is the the logical expression of the output of this circuit? [4]

iii. Simplify the logical expression in (ii). Explain your answer. [4]

iv. Draw the resulting simplified circuit. [2]

(b) A set of three sensors in a factory detects whether the pollution level it is outputting from an incinerator exceeds the safety limit. In which case the incinerator is shut down. An alarm A goes off if at least two the three sensors s_1, s_2 and s_3 detect a pollution level above the limit. Draw a logic circuit for the system showing the inputs s_1, s_2 and s_3 and the output put A.

[4]

(c) i. Name two properties a function has to satisfy to be a bijective function. [2]

ii. Consider the following function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 2x + 3$.

1. Show f is a bijective function. [4]
2. Find the inverse function f^{-1}. [2]

(d) Given the function $f : \mathbb{R} \to \mathbb{R}^+$ with $f(x) = 2^x$.

1. Find the inverse, f^{-1} [2]
2. Plot the curves of both function f and its inverse, f^{-1} in the same graph. [4]

Question 4 Graph Theory, Trees & Relations

(a) i. Give a definition of a bipartite graph. [2]

ii. Name an algorithm that is used to find the shortest path between vertices in a weighted graph. [2]

iii. Is it possible to draw a simple graph with a degree sequence, 5, 4, 3, 2, 2? If yes draw the graph and if no, explain why. [3]

iv. A graph G with 5 vertices: a, b, c, d, e has the following adjacency list:

$a : b,\ e$

$b : a,\ c,\ d$

$c : b,\ d$

$d : b,\ c,\ e$

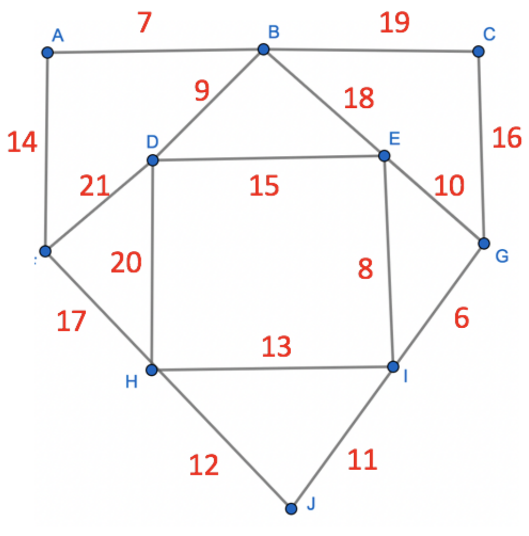
$e : d,\ a.$

1. Draw this graph, G. [2]
2. Write down the adjacency matrix, A_G, of the graph G. [2]

(b) i. What is the number of edges in a tree with n vertices? [2]

ii. Explain how to find the minimum spanning tree in a weighted graph using Kruskal's algorithm. [2]

iii. Given the following undirected weighted graph with 10 vertices, $A, B, \cdots J$. The number (weight) on each edge represent the distance, in kilometres, pairs of vertices.



1. Using Kruskal's find the the minimum spanning tree of this graph. [3]
2. What it the length of the resulting spanning tree? [2]

(c) Given S be the set of integers $\{5, 6, 7, 8, 9, 10\}$. Let $\mathcal{R}$ be a relation defined on S by the following condition such that,
for all $x, y \in S$, $x\mathcal{R}y$ if $(x + y) \mod 2 = 0$.

i. Draw the digraph of $\mathcal{R}$. [2]

ii. Say with reason whether or not $\mathcal{R}$ is

- reflexive;
- symmetric;
- anti-symmetric;
- transitive.

In the cases where the given property does not hold provide a counter example to justify this. [4]

iii. is $\mathcal{R}$ a partial order? Explain your answer. [1]

iv. is $\mathcal{R}$ an equivalence relation? If the answer is yes, write down the equivalence classes for this relation and if the answer is no, explain why. [3]