UNIVERSITY OF LONDON

GOLDSMITHS COLLEGE

Department of Computing

B. Sc. Examination 2019

IS51002E
Mathematical Modelling for Problem Solving

Duration: 3 hours

Date and time:

This paper is in two parts: part A and part B. You should answer ALL questions from part A and THREE questions from part B. Part A carries 40 marks, and each question from part B carries 20 marks. The marks for each part of a question are indicated at the end of the part in [.] brackets.

There are 100 marks available on this paper.

The use of calculators is allowed. Students are required to note the model of the calculators on the answer sheet.

Electronic calculators must not be programmed prior to the examination. Calculators which display graphics, text or algebraic equations are not allowed.

THIS PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Part A

Multiple choice

Question 1 Each question has one or more correct answers

(a) Let $A = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}\}$. Which of the following sets represent A using the inclusion rules? More than one answer may apply.

i. $\{2^{-n} : n \in \mathbb{N}$ **and** $n \leq 8\}$

ii. $\{2^{-n} : n \in \mathbb{Z}$ **and** $0 < n < 9\}$

iii. $\{\frac{1}{2n} : n \in \mathbb{Z}$ **and** $0 \leq n \leq 8\}$

iv. $\{\frac{1}{2n} : n \in \mathbb{Z}$ **and** $0 \leq n < 8\}$

[2]

(b) Let $S = \{a, b, c\}$, which one of the following sets represents $\mathcal{P}(S)$?

i. $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

ii. $\{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

iii. $\{\{a\}, \{b\}, \{c\}, \{a, b\}, a, b, \{b, c\}\}$

iv. $\{\{a\}, \{b\}, \{c\}\}$

[2]

(c) Let p and q and be two propositions where p means '**Sarah is not Happy**' and q means '**Sarah is reading a book**'. Which one of the following logical expressions is a correct formalisation of the following sentence:

Sarah is either happy or is reading a book, but not both

i. $\neg p \rightarrow q$

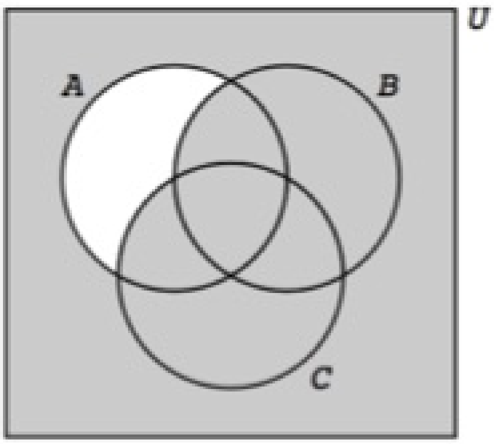
ii. $q \rightarrow \neg p$

iii. $\neg p \oplus q$

iv. $p \rightarrow \neg q$

[2]

(d) Let A, B and C be three subsets of the universal set U. Which one of the following sets correctly represents the shaded area in the following Venn diagram



.

i. $\bar{A} \vee B \vee C$

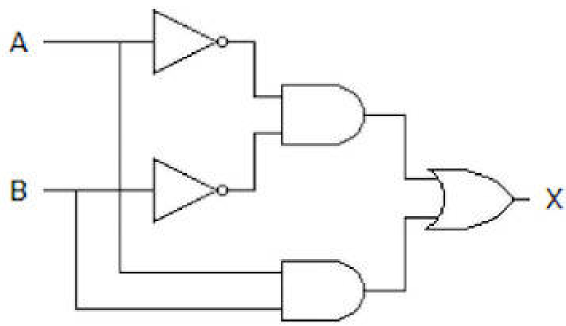
ii. $\bar{A} \vee (B \wedge C)$

iii. $\bar{A} \wedge B \wedge C$

iv. $\bar{A} \oplus (B \vee C)$

‘ [2]

(e) Which one is a correct output of the following logic network:



.

i. $(A \wedge B) \vee (\neg A \wedge \neg B)$

ii. $(A \wedge B) \vee (\neg A \wedge B)$

iii. $(A \wedge B) \vee (\ A \wedge \vee B)$

iv. $(A \vee B) \wedge (\neg A \vee \neg B)$

[2]

(f) Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a function where $f(x) = \log_{10} x$. Which one of the following is the inverse function of the function f?

i. $f^{-1}(x) = e^x$
ii. $f^{-1}(x) = 10^x$
iii. $f^{-1}(x) = \sqrt[10]{x}$
iv. $f^{-1}(x) = \frac{x}{10}$

[2]

(g) The following sequence $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \cdots$ is

i. arithmetic
ii. geometric
iii. neither geometric nor arithmetic
iv. both arithmetic and geometric

[2]

(h) Let p and q be two propositions. Which one of the following compound statements is equivalent to $p \vee \neg(p \wedge q)$

i. $(p \vee \neg p) \wedge (p \vee \neg q)$
ii. $\neg p \vee \neg q$
iii. T
iv. F

[2]

(i) Which of the following statements is/are **TRUE**? More than one answer might apply.

i. it is possible to draw a 5-regular graph with 5 vertices
ii. it is possible to draw 2-regular graph with 5 vertices
iii. the sum of the degree sequence of a graph is twice the number of edges in the graph
iv. the sum of the degree sequence of a graph is twice the number of vertices in the graph.

[2]

(j) The number of of edges in a complete graph with n vertices is

i. n-1
ii. n(n-1)
iii. n(n-1)/2
iv. 2n

[2]

(k) What is the decimal representation of 152_{16}?

i. 337_{10}
ii. 593_{10}
iii. 145_{10}
iv. 338_{10}

[2]

(l) What is the fractional representation of the recurring decimal in simplest form 5.981981...?

i. $\frac{101}{111}$
ii. $\frac{664}{111}$
iii. $\frac{909}{999}$
iv. $\frac{5909}{999}$

[2]

(m) $3\log_2 8$ is equal to:

i. 24
ii. 9
iii. $\log_2 24$
iv. is not defined

[2]

(n) $\log_{10} -1$ is equal to

i. 0
ii. -1
iii. -0.1
iv. is not defined

[2]

(o) The graph of 2^{x+1}:

i. has a x-intercept of 1
ii. has a y-intercept of 1
iii. passes through the point $(0, 2)$
iv. passes through the point $(2, 0)$

[2]

(p) Calculate the following limit: $\lim_{x \to 1} \frac{x^2-1}{x-1}$.

i. 2
ii. ∞
iii. $\frac{1}{2}$
iv. is not defined

[2]

(q) Given $y = \frac{1}{x}$

i. $\frac{dy}{dx} = \frac{1}{1}$
ii. $\frac{dy}{dx} = -\frac{1}{x^2}$
iii. $\frac{dy}{dx} = \ln x$
iv. $\frac{dy}{dx} = e^x$

[2]

(r) Convert the vector $(2, 2)$ in polar coordinates to cartesian coordinates

i. $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$
ii. $\begin{pmatrix} 2\sqrt{2} \\ \frac{\pi}{2} \end{pmatrix}$
iii. $\begin{pmatrix} 2\sqrt{2} \\ \frac{3\pi}{2} \end{pmatrix}$
iv. none of the above

[2]

(s) You are given vectors $\underline{u}=\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and $\underline{v}=\begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}$

$\underline{u} \times \underline{v}$ the cross product (vector product) of $\underline{u}$ and $\underline{v}$ is equal to

i. 3

ii. 4

iii. $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

iv. $\begin{pmatrix} 10 \\ -1 \\ 5 \end{pmatrix}$

[2]

(t) Find M^{-1}, the inverse of M where $M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

i. $\begin{pmatrix} \frac{1}{2} & 0 & -1 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

ii. $\begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

iii. is undefined

iv. none of the above

[2]

Part B

Question 2 Set, Logic & Sequences

(a) i. Describe the set A by the listing method.

$$A = \{r^{-2} : r \in \mathbb{Z} \textbf{ and } -1 < r \leq 4\}.$$

ii. Describe the set B using the set builder notation (inclusion rules) where $B = \{1, 3, 9, 15, 81, 243\}$.

[2]

(b) Let A and B and C be subsets of a universal set $\mathcal{U}$.

1. Draw a labelled Venn diagram depicting A, B, C in such a way that they divide $\mathcal{U}$ into 8 disjoint regions. [1]
2. The subset $X \subseteq \mathcal{U}$ is defined by the following membership table:

A	B	C	X
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

Shade the region X on your diagram. Describe the region you have shaded in set notation as simply as you can. [3]

(c) Let $A = \{x | x \in \mathbb{N} \textbf{ and } x < 20\}$. Let p and q be the following propositions concerning a positive integer n in A.

p : *' n is a prime number '*

q : *'n is less than 10 '.*

i. Express each of the three following compound propositions concerning positive integers symbolically by using p, q and appropriate logical symbols.

'n is a prime number if it is less than 10'

'n is a prime number only if it is less than 10'

'n is prime number or a positive integer less than 10, but not both'

[3]

ii. Find the truth set for the logical statement $q \to p$. [2]

iii. Write in words the contrapositive of the statement given symbolically by '$q \to p$'. [2]

(d) i. Express the following sum using the $\sum$ notation [1]

$$1 + 3 + 5 + 7 + ... + (2n + 1)$$

ii. Evaluate the following the following sum: [2]

$$\sum_{k=21}^{100} (2k - 1)$$

Hint: you might want to use the formula: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

iii. Given the following sequence: $u_1 = 1$ and $u_{n+1} = u_n + n + 1$ **for all** $n \geq 1$.

1. Calculate u_1, u_2. [1]
2. Prove by induction that: $u_n = \frac{n(n+1)}{2}$, **for all** $n \geq 1$. [3]

Question 3 Graphs, Trees & Relations

(a) i. Is it possible to construct a 3-regular graph with 5 vertices ? Explain your answer. [1]

ii. Is it possible to construct a simple graph with the degree sequence 5,4,3,2,2? Explain your answer. [1]

iii. Draw the two graphs with adjacency lists. [2]

- $a_1 : a_2, a_5$
- $a_2 : a_1, a_3, a_4, a_5$
- $a_3 : a_2, a_4, a_5$
- $a_4 : a_2, a_3, a_5$
- $a_5 : a_1, a_2, a_3, a_4$

and

- $b_1 : b_2, b_3, b_4, b_5$
- $b_2 : b_1, b_5$
- $b_3 : b_1, b_4, b_5$
- $b_4 : b_1, b_3, b_5$
- $b_5 : b_1, b_2, b_3, b_4$

1. Write down the degree sequence for each graph above. [2]
2. Are these graphs isomorphic? If so, show the correspondence between them. [3]

(b) i. How many vertices are there in a tree with n edges? [1]

ii. A binary search tree is designed to store an ordered list of 4000 records, numbered 1,2,3,...,3000 at its internal nodes.

Draw levels 0, 1 and 2 of this tree, showing which number record is stored at the root and at each of the nodes at level 1 and 2, making it clear which records are at each level and find the height of this tree? [4]

(c) Given S be the set of integers $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Let $\mathcal{R}$ be a relation defined on S by the following condition such that,
for all $x, y \in S$, xRy if $x - y \; mod \; 2 = 0$.

i. Draw the digraph of $\mathcal{R}$. [2]

ii. Show that $\mathcal{R}$ is an equivalence relation. [3]

iii. Write down the equivalence classes of $\mathcal{R}$. [1]

Question 4 Bases, Modular Arithmetic & Trigonometry

(a) i. Express the decimal number $(81.625)_{10}$ as a binary number. [2]

ii. Express the hexadecimal number $(1F4.E)_{16}$ as a decimal number. [1]

iii. Express the octal number $(173.16)_8$ as

(1) a binary number

(2) a hexadecimal number [3]

iv. Working in base 16 and showing all your working, compute the following:

$$(4AA2)_{16} + (394)_{16} - (1F92)_{16}$$

[2]

(b) i. Find the smallest positive integer modulo 13 that is congruent to

(1) 162

(2)1662 [2]

ii. Find the following

(1) the additive inverse of 11 modulo 13

(2) the multiplicative inverse of 11 modulo 13 [2]

(c) i. Triangle ABC has side $a = 5cm$, side $b = 6.2cm$ and angle $B = 0.873$ radians Find

(1) the size of angle A

(2) the size of angle C

(3) the length of side c [3]

ii. Given $f(x) = \sin(x + \frac{\pi}{4})$ and $g(x) = 2\cos 2x$

(1) Find the amplitude, frequency and period for

- $f(x)$
- $g(x)$

[3]

(2) By plotting the graphs of $f(x)$, or otherwise, find all the values of x between $-\pi$ and π for which $\sin(x + \frac{\pi}{4}) = 0.5$. [2]

Question 5 Matrices & Complex Numbers

(a) Let A be a 3x3 homogeneous transformation matrix corresponding to a scaling of the y-coordinates only by a factor of 3. Let B be a 3x3 homogeneous transformation matrix corresponding to a translation of the x and y-coordinates by -1 and 1 respectively. Let C be a 3x3 homogeneous transformation matrix corresponding to an anti-clockwise rotation of π about the z-axis.

i. Find the matrices A, B and C. [3]

ii. How would the transformation represented by the matrix C transform the following three points which represent a triangle in the Cartesian space: (1,0), (0,1) and (2,1)? [2]

iii. Find the inverse matrices A^{-1} and B^{-1}. [2]

iv. Find the single matrix D which represents the transformation represented by matrix B followed by transformation represented by matrix A. [2]

v. Find the inverse matrix D^{-1} [2]

(b) Given the complex numbers $z_1 = 2 + i$ and $z_2 = 3 - i$.

i. represent z_1 and z_2 on an Argand diagram [2]

ii. Find

(2) $z_1 - z_2$

(3) $z_1 \times z_2$

(4) $\overline{z_2}$ [3]

iii. Convert z_1

(1) to polar form

(2) to exponential form [2]

iv. Find ${z_1}^3$, give your answer in exponential form [2]

Question 6 Functions, Graph Sketching & Probability

(a) i. Find numerical values for the following:

(1) $\log_2(\frac{1}{4})$

(2) $\log_4(\frac{1}{2})$ [2]

ii. Sketch the graph of

(1) $f(x) = 3^{-x}$

(2) $g(x) = \log_3 x - 1$ [3]

iii. Find the inverse functions:

(1) $f^{-1}(x)$

(2) $g^{-1}(x)$ [2]

(b) Given $\underline{v}_1 = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$ and $\underline{v}_2 = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$

i. rewrite $\underline{v}_1$ in terms of standard unit vectors.

ii. Find the magnitudes of $\underline{v}_1$ and $\underline{v}_2$.

iii. Find the dot product $\underline{v}_1 \cdot \underline{v}_2$.

iv. Hence find the angle between $\underline{v}_1$ and $\underline{v}_2$.

[7]

(c) Ten balls - five green, three blue, two yellow - are placed in a hat. Two balls are drawn, one after the other, without replacement.

i. What is the probability that both balls are green?

ii. What is the probability that none of the balls are green?

iii. What is the probability that at least one will be green?

[6]