**UNIVERSITY OF LONDON**

**GOLDSMITHS COLLEGE**

**Department of Computing**

**B. Sc. Examination 2015**

**IS51002D**
**Mathematical Modelling for Problem Solving**

**Duration: 3 hours**

**Date and time:**

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*There are ten questions in this paper. You should attempt all questions. Full marks will be awarded for complete answers to a total of ten questions. Each question carries 10 marks. The marks for each part of a question are indicated at the end of the part in* [.] *brackets.*

*There are 100 marks available on this paper.*

**THIS PAPER MUST NOT BE REMOVED**
**FROM THE EXAMINATION ROOM**

## Question 1

(a) The first 16 integers $\geq 0$ can be represented by 4-bit binary strings.

i. List these integers in hexadecimal, together with their binary equivalents.

ii. Find the hexadecimal equivalent of the binary numeral $100101.01$ and find the binary equivalent of the hexadecimal numeral $59.A$

[4]

(b) Working in the binary system compute the following operation, showing all your working:

$$(10110)_2 + (11011)_2 - (111)_2$$

[2]

(c) i. Define what is meant by an irrational number.

ii. Showing all your working, express the repeating decimal $0.270270$..... as a fraction in its simplest terms.

[4]

## Question 2

(a) Let $A = \{2n : n \in \mathbb{Z}^+\}$ and $B = \{3, 6, 9, 12, ...\}$ be two sets of numbers.

i. Describe the set $A$ by the listing method.

ii. Describe the set $B$ by the rules of inclusion method.

iii. Find the two sets $A \cap B$ and $A - B$, by the listing method. [5]

(b) Let $P,\ Q$ and $R$ be subsets of a universal set $\mathcal{U}$.

i. Construct a membership table for the set $X = P' \cup (Q \cap R)$.

ii. Draw a labelled Venn diagram showing $P, Q,$ and $R$ intersecting in the most general way.

iii. Shade the region $X$ on your diagram.

iv. Is the set $P' \cap R \subseteq X$? Justify your answer. [5]

## Question 3

(a) Let $n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and let $p, q$ be the following propositions concerning the integer $n$.

$$p: \; n \text{ is odd} \qquad q: \; n < 5.$$

By drawing up the appropriate truth table find the truth set for each of the propositions $p \vee \neg q$; $\neg(q \rightarrow p)$.

[4]

(b) Construct and draw a logic network that accepts as inputs $p$ and $q$, which may independently have the value 0 or 1, and gives as final output $(p \wedge q) \vee \neg q$. Label all the gates appropriately and also give labels to show the output from each gate.

i. Construct a logic table to show the value of the output corresponding to each combination of values (0 or 1) for the inputs $p$ and $q$.

ii. Show that $(p \wedge q) \vee \neg q$ is equivalent to $q \rightarrow p$. [6]

## Question 4

(a) Given the following formulae:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \text{and} \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Write the following sums in $\sum$-notation and evaluate them.

i. $50 + 51 + 52 + ... + 100$

ii. $1^2 + 2^2 + 3^2 + ... + 40^2$

[4]

(b) A sequence is defined by the recurrence relation

$$u_{n+1} = 5u_n + 4 \text{ and initial term } u_1 = 4.$$

i. Calculate $u_2$ and $u_3$, showing your working.

ii. Prove by induction that

$$u_n = 5^n - 1 \text{ for all } n \geq 1.$$

[6]

## Question 5

(a) What properties should a graph have in order for it to be:

i. a simple graph;
ii. a complete grqph

[2]

(b) Say why is it imposible to construct a simple graph with the following degree sequence:

$$5, 4, 3, 3, 2$$

[1]

(c) i. Draw a complete graph $K_5$.
ii. State the number of edges in $K_5$.
iii. Find the number of different paths of length 2 from $v_1$ to $v_2$ in $K_5$ (not counting cycles of length 2).
iv. Find an expression for the number of edges in $K_n$.
v. Find an expression for the number of paths of length 2 from $v_1$ to $v_2$ in $K_n$.

[7]

## Question 6

(a) Let $G$ be the simple graph with vertex set $V(G) = \{a, b, c, d, e\}$ and adjacency matrix

$$\mathbf{A} = \begin{array}{c} \\ \text{a} \\ \text{b} \\ \text{c} \\ \text{d} \\ \text{e} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{a} & \text{b} & \text{c} & \text{d} & \text{e} \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

i. Say how the number of edges in $G$ is related to the entries in the adjacency matrix $\mathbf{A}$ and calculate this number.
ii. Draw $G$.
iii. Find a spanning tree $T_1$ for $G$ and give its degree sequence.
iv. Find a spanning tree $T_2$ for $G$ which is **not** isomorphic to $T_1$ and give a reason why it is not isomorphic.

[6]

(b) Draw a binary search tree to store 14 records and find the maximum number of comparison needed to find any existing records. [4]

**Question 7**

Let $S$ be the set of integers $\{1, 2, 5, 6, 7, 8, 9\}$.

(a) Let $\mathcal{R}_1$ be a relation defined on $S$ by the following condition such that, for all $x, y \in S$, $x\mathcal{R}_1 y$ if $x \leq y$ .

i. Draw the digraph of $\mathcal{R}_1$.

ii. Show that $\mathcal{R}_1$ is a total order.

[5]

(b) Another relation, $\mathcal{R}_2$, is defined on $S$ as follows:

$$\text{for all} x, y \in S,\ x\mathcal{R}_2 Y \text{ if } (x + y)\ mod\ 2\ =\ 0$$

i. Draw the digraph of this relation on $S$.

ii. Explain why this relation is an equivalence relation but not a partial order.

iii. Write down the equivalence classes for this relation. [5]

**Question 8**

(a) Which of the following homogeneous coordinates (2,6,3), (4,6,2), (2,4,1) and (8,12,4) represent the point (2, 3)? [1]

(b) Let A be a 3x3 homogeneous matrix transformation corresponding to an anti-clockwise rotation about the z-axis by an angle $\frac{\pi}{2}$ and let B be a 3x3 homogeneous matrix transformation to scale the x and y coordinates by a factor 3 and 2 respectively

i. Write down A, B

ii. Find the single homogeneous matrix, C, which represents transformation represented by the matrix A followed by transformation represented by the matrix B.

iii. How would the combined transformation represented by the matrix C transform the following three points which represent a triangle in the Cartesian space: (0,0), (-1,1) and (1,1)?

iv. Find the matrix $A^{-1}$ [9]

**Question 9**

(a) In an experiment a coin is tossed three times and each time it is noted whether the coin comes up heads (H) or tails (T). The final result is recorded as an ordered triple, such as (H,H,T). Let $A$ be the event that the last toss comes up as a tail and $B$ be the event that there is only one tail in the triple.

i. Draw a rooted tree to model this process.

ii. Calculate the probabilities of the events $A,\ B,\ A \cap B$ and $A \cup B.$

iii. Are $A$ and $B$ independent events? Justify your answer.

[7]

(b) In a class of 60 students in how many different ways can

i. a group of 3 students be chosen?

ii. a first, second and third prize be awarded in a class competition if each student can receive at most one prize? [3]

**Question 10**

(a) A function $f : X \to Y$, where $X = \{p, q, r, s\}$ and $Y = \{1, 2, 3, 4, 5\}$, is given by the subset of $X \times Y$, $\{(q, 3), (r, 3), (p, 5), (s, 2)\}$.

i. Show $f$ as an arrow diagram.

ii. Say why $f$ does not have the ***one-to-one*** property and why $f$ does not have the ***onto*** property, giving a specific counter example in each case.

[3]

(b) i. Differentiate the following function with respect to x:

$$f(x) = (1 + x)e^x$$

ii. Find the following limits:

$$\lim_{x\to+\infty} \frac{x^2 + x - 6}{x - 2} \text{ and } \lim_{x\to 2} \frac{x^2 + x - 6}{x - 2}$$

iii. Show that the function

$$f(x) = \frac{1}{x} + x^2$$

has exactly one stationary point. Determine whether the stationary point is a local maximum, a local minimum, or a point of inflection.

[7]