UNIVERSITY OF LONDON

GOLDSMITHS COLLEGE

Department of Computing

B. Sc. Examination 2014

IS51002C
Mathematical Modelling for Problem Solving

Duration: 2 hours 15 minutes

Date and time:

There are Six questions in this paper. You should answer no more than THREE questions. Full marks will be awarded for complete answers to a total of THREE questions. Each question carries 25 marks. The marks for each part of a question are indicated at the end of the part in [.] *brackets.*

There are 75 marks available on this paper.

THIS PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Question 1

(a) i. Express the binary number $(1000111.11)_2$ as a decimal number.

[2]

ii. Express the binary number $(1011011.011)_2$ as a hexadecimal number.

[2]

iii. Express the hexadecimal number $(7C.2)_{16}$ as a binary number.

[2]

iv. Working in base 16 perform the following addition. Show all your working:

$$4B3 + 92D$$

[2]

v. Working in the binary system compute the following sums, showing all your working:

$$1100111 + 1000011 \qquad 11001100 - 1101011$$

[2]

(b) i. Showing all your working, express the recurring decimal $0.4545...$ as a fraction in its lowest form.

[3]

ii. Using the method of repeated division, or otherwise, convert the decimal number 4768 to base 8, showing all your working.

[2]

(c) i. Let A, B and C be subsets of a universal set $\mathcal{U}$.

1. Construct the membership table for $(A \cup B)' \cap C$.
2. Use the membership table or otherwise to Show that:

$$(A \cup B)' \cap C = (A' \cap B') \cap C.$$

[6]

ii. Given the sets

$$\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$
$$A = \{1, 2, 5, 6, 8\}$$
$$B = \{3, 5, 7, 8\}$$
$$C = \{5, 6, 7, 8, 9\}.$$

1. List separately the elements of $A' \cap B$ and $A' \cup C$.
2. Describe, as simply as you can in terms of set operations on A, B and C, the sets $\{5, 8\}$ and {1,2,7,9}.

[4]

Question 2

(a) Let n be an element of the set $\{$10,11, 12,13,14,15,16,17,19,20$\}$ and p and q be the following statements about n:

p: $n \leq 15$

q: n is odd.

i. Express each of the three following compound propositions symbolically by using p, q and appropriate logical symbols.

$n{>}15$ and n is even.

if $n{\leq}15$ then n is even.

$n{\leq}15$ or n is odd, but not both.

[3]

ii. Draw up the truth tables for the following statements and find the values of n for which they are true:

$$p \vee q; \qquad \neg p \wedge q$$

[4]

iii. Use the truth table to find a statement that is logically equivalent to $\neg p \rightarrow q$.

[3]

(b) Let p and q be two propositions defined in (a)

i. Write the contrapositive of the following statement:

if $n > 15$ ithen n is even

[3]

ii. Rewrite the result in (i) using logical symbols.

[2]

(c) i. Draw a logic network that accepts independent inputs p and q and gives as output

$$p \wedge (\neg p \vee q).$$

[4]

ii. Construct the truth table for this output.

[3]

iii. Hence, or otherwise, find a simpler expression that is logically equivalent to the final output.

[3]

Question 3

(a) i. State the condition to be satisfied in order for a function to have an inverse.

[1]

ii. Let $A\{1,2,3,4,5\}$ and $f : \mathbb{A} \to \mathbb{Z}$ with $f(x) = \lceil \frac{x^2-1}{4} \rceil$

1. Find $f(2)$ and the ancestor of 0.
2. Find the range of f.
3. Is f invertible? Justify your answer.

[4]

iii. Let $g : \mathbb{Z} \to \mathbb{Z}$ with $g(x) = \lceil \frac{x-1}{4} \rceil$

1. Find $g(4)$ and the ancestors of 0.
2. Find the range of g.
3. Is g invertible? Justify your answer.

[5]

(b) Let $f : \mathbb{R} \to [1, \infty[$ with $f(x) = x^2 + 1$

i. Plot the function f.

[2]

ii. Show that f is not invertible.

[3]

(c) i. Let $g : \mathbb{Z} \to \mathbb{Z}$ with $g(x) = 3x + 5$

1. Is g a one to one function? Justify your answer.
2. Is g an onto function? Justify your answer.
3. Is g an invertible function? Justify your answer.

[5]

ii. Let $h : \mathbb{R} \to \mathbb{R}$ with $g(x) = 3x + 5$

1. Show that h is an invertible function.
2. Find h^{-1}, the inverse function of h.

[5]

Question 4

(a) i. Draw a binary tree to store a list of 13 records.

[6]

ii. What is the maximum number of comparisons that would have to be made in order to locate an existing record from this list of 13 records?

[2]

iii. Find the height of binary search tree to store a list of 4000. records numered 1,2,...4000 at its internal nodes.

[2]

(b) i. Use the formula $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ to find a formula for $s_n = \sum_{k=1}^{n}(5k+1)$ in terms of n. Use this formula to find this sum when $n = 10$.

[3]

ii. Write the following expression in $\sum$ notation using appropriate limits and calculate its value.

$$5 + 10 + 15 + 20 + \cdots + 95 + 100$$

[2]

(c) i. Given the following sequence

$$1, 2, , 4, 8, 16 \cdots$$

1. Is this sequence arithmetic of geometric? If you identify it as arithmetic, specify the common difference d. If you identify it as geometric, specify the common ratio r.
2. In terms of n find an expression for the sum of the first n terms of this sequence.
3. Find the sum of the first 10 terms.

[5]

ii. Let u_n be the sequence of numbers defined by

$$u_1 = 0; \text{ and}$$
$$u_{n+1} = 2u_n + 1$$

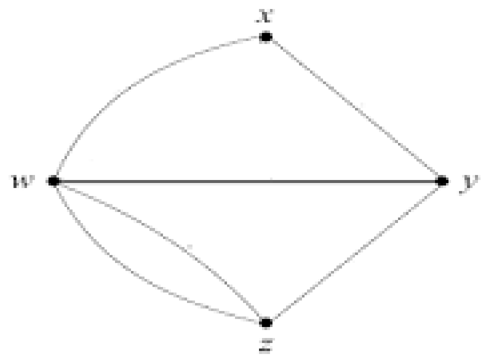
1. Calculate u_2 and u_3.
2. Prove by induction that:

$$u_n = 2^{n-1} - 1 \qquad \text{for all } n \geq 1$$

[5]

Question 5

(a) Consider the following graph, G, with 4 vertices x, y, z and w.



i. Find the vertices adjacent to z.

[1]

ii. Find the degree sequence of G.

[2]

iii. Find 2 non-isomorphic spanning trees of G.

[3]

iv. Let A be the adjacency matrix of G. Write down A.

[2]

v. What information does the sum of all the elements in the matrix A tell you about G?

[2]

(b) i. Given the vector $\vec{v} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\vec{i} + 2\vec{j}$

1. Find the magnitude of $\vec{v}$.
2. Find the angle between $\vec{v}$ and the x-axis ($\vec{i}$).

[3]

ii. Which of the following homogeneous coordinates (2,6,2), (2,6,4), (1,3,1), (1,3,2), and (4,12,8) represent the point $(\frac{1}{2}, \frac{3}{2})$?

[2]

(c) i. The matrix of anti-clockwise rotation about the z-axis with angle θ is

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1. Find the transformation matrices A and B, corresponding to an anti-clockwise rotations about the z-axis by an angle $\frac{\pi}{2}$ and π respectively.
2. How does A and B transform a point $p(x, y)$?
3. Write B in terms of A.

[5]

ii. The following three points form a triangle in the Euclidean space.

$$(0, 0), (0, 2), (2, 0)$$

Show how this triangle is transformed if the following transformation is applied.

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[3]

iii. Find $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$

[2]

Question 6

(a) Let G be a simple graph with vertex set $V(G) = \{v_1,\ v_2,\ v_3,\ v_4,\ v_5\}$ and adjacency lists as follows:

$$v_1{:}v_2\ v_3\ v_4$$
$$v_2{:}v_1\ v_3\ v_4\ v_5$$
$$v_3{:}v_1\ v_2\ v_4$$
$$v_4{:}v_1\ v_2\ v_3.$$
$$v_5{:}v_2$$

i. List the degree sequence of G.

[2]

ii. State the relation between the degree sequence and the total number of edges in G. Hence, find the number if edges in G.

[2]

iii. Draw the graph of G.

[2]

iv. Find two distinct paths of length 3, starting at v_3 and ending at v_4.

[2]

v. Find a 4 cycle in G.

[2]

(b) In the following cases either construct a graph with the specified properties or say why it is not possible to do so.

i. A graph with degree sequence 3,2,2,1.

[3]

ii. A simple graph with degree sequence 5,4,3,2,2.

[2]

(c) Let S be the set $\{2, 3, 4, 5, 6, 7\}$ and a relation $\mathcal{R}$ is defined between the elements of S by

"x is related to y if $x\ mod\ 2 = y\ mod\ 2$".

i. Draw the relationship digraph.

[2]

ii. Determine whether or not $\mathcal{R}$ is reflexive, symmetric, anti-symmteric or transitive. In cases where one of these properties does not hold give an example to show that it does not hold.

[4]

iii. State, with reason, whether $\mathcal{R}$ is a partial order or not.

[1]

iv. State with reason, whether $\mathcal{R}$ is an equivalence relation. If the answer is yes, find the equivalence classes.

[3]