UNIVERSITY OF LONDON

GOLDSMITHS' COLLEGE

B. Sc. Examination 2003

STATISTICS

ST53004A Stochastic Processes

Duration: 2 hours 15 minutes

Date and time:

Answer THREE questions.
Full marks will be awarded for complete answers to THREE questions.
There are 60 marks available on this paper.
A Formula Sheet is attached to the end of this examination paper.
Electronic calculators may be used. The make and model should be specified on the script. The calculator must not be programmed prior to the examination. Calculators which display graphics, text or algebraic equations are not allowed.
NOTE: Full details of all calculations are to be shown; pre-programmed statistical tests and procedures on a calculator, apart from mean and standard deviation, must not be used.
WHITE, YEATS & SKIPWORTH: Tables for Statisticians to be provided.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Question 1 A gambler with initial capital £k plays against an opponent with initial capital £$(a-k)$. At each turn the gambler wins £1 with probability $p > 0$, or loses £1 with probability $q > 0$ $(p + q = 1)$. If he reaches £a (pre-specified), he retires as his opponent is ruined. If the gambler is ruined, that is with capital £0, he does not leave, but he continues playing, and a rich uncle pays any losses he incurs whilst ruined.

Find a difference equation for the expected number of turns d_k before he retires, and solve the equation in the cases $p \neq q$ and $p = q = 1/2$. [18]

Discuss briefly what happens to these results as $a \to \infty$ in the case $p > 1/2$. [2]

Question 2 (a) For a Markov Chain with transition matrix $P = (p_{ij})$, define what it means

(i) for a state i to intercommunicate with a state j [2]

(ii) for a set of states C to be closed [2]

(iii) for a state i to be persistent (recurrent) [2]

(b) A Markov Chain with 6 states has transition matrix given by

$$\begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$$

(i) Find the closed sets of the chain, and hence classify all states as transient or persistent (recurrent). [6]

(ii) Considering each closed set as a chain, write down its transition matrix, and find the stationary distribution. Hence calculate the mean recurrence time for all persistent (recurrent) states. [8]

Question 3 Two boxes each contain 4 balls. There are 4 red balls and 4 blue balls. At each step a ball is selected at random from each box and the two balls are exchanged (so there are still 4 balls in each box). Let X_n be the number of red balls in the first box after n steps.

(a) Find the transition matrix for the Markov chain $\{X_n\}$ justifying your answer. [8]

(b) Hence find the stationary distribution of $\{X_n\}$. [8]

(c) If there are initially 2 red balls in the first box, find the probability distribution of X_2. [4]

Question 4 A certain type of cell follows a linear birth process with parameter $\lambda > 0$. Find the forward system of equations. [6]

Hence derive a partial differential equation for the probability generating function $G(z, t)$ of the number of cells $N(t)$ at time t. [4]

Given that there was a single cell at $t = 0$, show that

$$G(z,t) = ze^{-\lambda t}[1 - z + ze^{-\lambda t}]^{-1}. \quad [4]$$

After the colony has been growing for a time t, growth stops and then each cell decays independently with death occurring after an exponentially distributed time interval of mean μ^{-1}.

Show that after a further time s of decaying, the probability that the colony is extinct is

$$(e^{\mu s} - 1)(e^{\lambda t} + e^{\mu s} - 1)^{-1}. \quad [6]$$

Question 5 A bank has a single queue with k servers. Customers arrive according to a Poisson process of rate α and the service time is independent for each customer and exponentially distributed with mean β^{-1}.

(a) Show that an equilibrium exists if $\rho < k$, where $\rho = \alpha/\beta$ is the traffic intensity for a single server queue (you may quote but need not prove a theorem relating to the general birth-death process). [6]

(b) Write down the equilibrium distribution in this case in terms of ρ. [5]

(c) For three servers, show that the mean queue size is

$$\frac{\rho}{1 - \rho/3} \frac{6 + 2\rho - \rho^2/3}{6 + 4\rho + \rho^2}.$$

(HINT: $\sum_{m=1}^{\infty} ma^m = a/(1-a)^2$, when $|a| < 1$.) [5]

(d) In the case $\rho = 1$, find the efficiency gain for a single queue using this criterion as opposed to the queueing strategy where there is an independent queue for each server, with no swaps allowed. Mention any assumptions you make. [4]