UNIVERSITY OF LONDON

GOLDSMITHS COLLEGE

B. Sc. Examination 2003

MATHEMATICS

MT53010A (M334) Modern Analysis

Duration: 2 hours 15 minutes

Do not attempt more than FOUR questions on this paper.

Full marks will be awarded for complete answers to FOUR questions.

Electronic calculators may be used. The make and model should be specified on the script. The calculator must not be programmed prior to the examination. Calculators which display graphics, text or algebraic equations are not allowed.

THIS EXAMINATION PAPER MUST NOT BE

REMOVED FROM THE EXAMINATION ROOM

Question 1 (a)

(i) Let $f(t) = t/(1 + t)$. Show that $f(t)$ is monotone increasing on $[0, \infty)$. Deduce that for all real numbers a and b, $f(|a + b|) \le f(|a| + |b|)$. Hence or otherwise, prove

$$|a + b|/(1 + |a + b|) \le |a|/(1 + |a|) + |b|/(1 + |b|).$$

[3]

(ii) Let $\mathbf{C}^{\mathbf{N}} = \{(x_n)_n : x_n \in \mathbf{C}, n \in \mathbf{N}\}$ be the set of sequences of complex numbers and let $d((x_n)_n, (y_n)_n) = \Sigma_{n=0}^{\infty} |x_n - y_n|/(2^n(1 + |x_n - y_n|))$. Prove that $(\mathbf{C}^{\mathbf{N}}, d)$ is a metric space.

[6]

(b) Suppose that (X, d_0) is a compact metric space.

(i) Prove that (X, d_0) is complete.

[3]

(ii) Let $(C(X), d)$ be the metric space of continuous real–valued functions on X, where $d(f, g) = sup\{| f(x) - g(x)| : x \in X\}$. Prove that $(C(X), d)$ is complete.

[6]

(c) Let $l_1 = \{(x_n)_n : x_n \in \mathbf{C}, n \in \mathbf{N}$, and $\Sigma_{n \in \mathbf{N}} |x_n| < \infty\}$, and let $d((x_n)_n, (y_n)_n) = \Sigma_{n \in \mathbf{N}} |x_n - y_n|$. Let $F = \{(x_n)_n \in l_1 : \{n \in \mathbf{N} : x_n \neq 0\}$ is finite$\}$, i.e. elements of F are convergent complex sequences almost all of whose terms are zero. For each $n \in \mathbf{N}$, define an element $\mathbf{x}^{(n)} = (\mathbf{x}^{(n)}{}_k)_k \in l_1$ as follows:

$$\mathbf{x}^{(n)}{}_k = 1/k^2 \text{ for } k = 1, 2, ..., n, \text{ and } \mathbf{x}^{(n)}{}_k = 0 \text{ for } k = n + 1, n + 2, ...$$

Show that the sequence $(\mathbf{x}^{(n)})_n$ is a Cauchy sequence but does not converge to any point in F.

[7]

Question 2 (a) Let (X, d) be a metric space and suppose that f is a function from X into X.

(i) What does it mean to say that f is a *contraction*?

[2]

(ii) Suppose that X is complete and, for some natural number n, $f^{\,n}$ is a contraction, where $f^{\,n}$ is the composition $f \circ f \circ ... \circ f$ of f (n times), and $f^{\,1} = f$. Prove that f has a unique fixed point.

[4]

(b) Let (C[a, b], d) be the metric space of continuous real-valued functions on the closed interval [a, b], where $d(f, g) = sup\{|f(x) - g(x)| : x \in [a, b]\}$. Suppose that $v(t)$ is a continuous real–valued function on [a, b] and $k(t, y)$ is a continuous real–valued function on the closed triangular region in $\mathbf{R}^2$ whose vertices are (a, a), (a, b) and (b, b). Let T : C[a, b] $\to$ C[a, b] be the function defined by

$$\mathrm{T}(g)(t) = \nu(t) + \int_a^t k(t, y)\, g(y)\, dy.$$

(i) Show there exists a real number c such that for all $t \in [a, b]$

$$|\mathrm{T}(g)(t) - \mathrm{T}(h)(t))| \leq c\,(t - a)\, d(g, h).$$

Deduce that $d(\mathrm{T}(g), \mathrm{T}(h)) \leq c\,(b - a)\, d(g, h)$.

[3]

(ii) Prove by induction on n $\in$ **N** that for all $t \in [a, b]$

$$|\mathrm{T}^{\mathrm{n}}(g)(t) - \mathrm{T}^{\mathrm{n}}(h)(t))| \leq c^{\mathrm{n}}\,(t - a)^{\mathrm{n}}\, d(g, h) / \mathrm{n}!.$$

Deduce that $d(\mathrm{T}^{\mathrm{n}}(g), \mathrm{T}^{\mathrm{n}}(h)) \leq c^{\mathrm{n}}\,(b - a)^{\mathrm{n}}\, d(g, h) / \mathrm{n}!$.

[6]

(iii) Hence show that for some natural number n, T^{n} is a contraction. Deduce that T has a unique fixed point, explaining carefully your argument.

[6]

(iv) Prove that the Volterra Integral equation $\nu(t) = g(t) - \int_a^t k(t, y)\, g(y)\, dy$ has a unique solution $g(t) \in$ C{a, b].

[4]

Question 3 (a)

(i) Suppose that (X, d) is a complete metric space and $\{F_n : n \in \mathbf{N}\}$ is a sequence of closed non–empty sets in X such that $F_{n+1} \subseteq F_n$, and F_0 is compact. Prove that the intersection $\cap_{n \in \mathbf{N}} F_n$ is non–empty.

[5]

(ii) Define a Cantor set C as follows: $C_0 = [0, 1]$, and for $n \in \mathbf{N}$, C_{n+1} is obtained from C_n by dividing C_n into five equal closed intervals and deleting the second and fourth of these intervals. (So, e.g. $C_1 = [0, 1/5] \cup [2/5, 3/5] \cup [4/5, 1]$.) Let $C = \cap_{n \in \mathbf{N}} C_n$. Show that C is closed and non–empty.

[3]

(b) Suppose that (X, d) is a complete metric space.

(i) Define carefully the *Hausdorff metric h* on the family $H(X)$ of compact subsets of X.

[3]

(ii) Explain carefully why the Cantor set C (defined above) belongs to $H([0, 1])$.

[2]

(c)

(i) Suppose that (X, d) is a complete metric space and let $(f_1, ..., f_n)$ be an iterated function system in X realizing the ratio list $(r_1, ..., r_n)$. Defining a suitable contraction mapping on $H(X)$, prove fully that there is a unique attractor of $(f_1, ..., f_n)$.

[7]

(ii) Write down an iterated function system and ratio list for the Cantor set C (defined above) and explain why C is the unique attractor of your system. What is the similarity dimension of C?

[5]

Question 4 (a)

(i) State the Cauchy–Schwartz inequality for n–tuples $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$ of complex numbers.

[2]

(ii) What does it mean to say that the pair $(X, \langle\ \rangle)$ is an *inner product space* over a field $\mathbf{F}$ of scalars? Prove that $l_2 = \{(x_n)_n : x_n \in \mathbf{C}, n \in \mathbf{N}, \text{ and } \Sigma_{n \in \mathbf{N}} |x_n|^2 < \infty\}\}$ is a complex inner product space.

[5]

(iii) Suppose that $(X, \langle\ \rangle)$ is an inner product space. Stating precisely any inequalities you use, show that for all $x, y \in X$,

$$\|x + y\| \leq \|x\| + \|y\|.$$

Deduce $\|x + y\| = \|x\| + \|y\|$ if and only if $y = 0$ or $x = cy$ for some non–negative real number c.

[5]

(b)

(i) Suppose that $\{e_n : n \in \mathbf{N}\}$ is an orthonormal sequence in a Hilbert space H. Prove that for every $x \in H$, the series $\Sigma_{n=1}^{\infty} \langle x, e_n\rangle e_n$ converges in H. Formulate carefully any ancillary results to which you appeal.

[7]

(ii) Let $x_i(t) = t^i \in C[-1, 1]$ for i = 1, 2, ... Let $\langle x, y\rangle = \int_{-1}^{1} x(t)\, y(t)\, dt$. Find three orthonormal elements from the first three terms of the sequence $\{x_1(t), x_2(t), x_3(t), ...\}$.

[6]

Question 5 (a) Let H be a Hilbert space.

(i) Suppose that M is a subspace of H. What is the orthogonal complement of M?

[2]

(ii) Let $f : H \to \mathbf{F}$ be a continuous linear function into the field $\mathbf{F}$ of scalars. Prove that there exists a unique element $u \in H$ such that $f(x) = \langle x, u \rangle$ for all $x \in H$. Deduce that $\|f\| = \|u\|$.

[8]

(b) Let H be a Hilbert space and let $\mathrm{T} : H \to H$ be a bounded linear operator.

(i) Prove that there exists a unique bounded linear operator $\mathrm{T}^* : H \to H$ such that

$$\text{for all } x, y \in H, \langle \mathrm{T}x, y \rangle = \langle x, \mathrm{T}^*y \rangle .$$

[6]

(ii) Show that $\|\mathrm{T}^*\| = \|\mathrm{T}\|$.

[3]

(iii) If T is positive, show that $|\langle \mathrm{T}x, y \rangle|^2 \leq \langle \mathrm{T}x, x \rangle \langle \mathrm{T}y, y \rangle$ for all $x, y \in H$.

[6]

END OF EXAMINATION