UNIVERSITY OF LONDON

GOLDSMITHS COLLEGE

B. Sc. Examination 2003

MATHEMATICS

MT53007A(M331) Graph Theory

Duration: 2hours 15minutes

Date and time:

There are five questions on this examination paper.
Do not attempt more than FOUR questions.
Full marks will be awarded for complete answers to FOUR questions.

Electronic calculators may be used. The make and model should be specified on the script. The calculator must not be programmed prior to the examination. Calculators which display graphics, text or algebraic equations are not allowed.

BEGIN EACH QUESTION ON A NEW PAGE and number the question and parts.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Question 1

(a) Describe a basic tree growing algorithm for constructing a maximal tree that is a subgraph of a given graph G and contains a particular vertex v_1. [3]

(b) Explain how the algorithm you described in part (a) would be modified to label the vertices of a given tree to produce

(i) a breadth-first search tree, rooted at v_1, and

(ii) a depth-first search tree rooted at v_1. [3]

(c) Suppose G is a tree such that: it has a unique breadth-first-search labelling rooted at v_1; it has a unique depth-first-search labelling rooted at v_1; and these two labellings are identical. What can you deduce about G? Explain your answer. [4]

(d) Explain how you would use the algorithm you described in part (a) to find the strongly connected components of a digraph. [4]

(e) Let D be the following digraph.

Using the algorithm you described in part (d), find the strongly connected components of D. You should draw clearly any trees you use at each stage. [9]

(f) Suggest a new arc to add to D to make the resulting digraph strongly connected. Give a brief reason for your answer. [2]

Question 2

(a) Suppose T is a tree in which all the vertices have degree one or four. Let n_1 be the number of vertices of degree 1 and let n_4 be the number of vertices of degree 4, so that $|V(T)| = n_1 + n_4$. Using the Handshaking Lemma or otherwise, show that $n_1 = 2(n_4 + 1)$. [5]

Consider the following network, N.

(b) Use Kruskal's algorithm to find a minimum weight spanning tree for N. Indicate the order in which you chose the edges and finish by clearly stating the weight of your tree. [5]

(c) Use Dijkstra's algorithm to grow a tree containing the shortest paths from v_1 to all other vertices in N. [9]

(d) By commenting on the cycle $v_1v_2v_3v_4v_1$ explain why the trees you obtained in parts (b) and (c) were different. [3]

(e) Explain carefully why the shortest path from v_1 to v_{11} that passes through v_{10} has weight 24. [3]

Question 3

(a) Prove that if a digraph, D, has k arc-disjoint xy-paths, then $d^+(U) \geq k$ for all $U \subset V(D)$ with $x \in U$ and $y \in V(D) - U$. (This is the "necessity" part of Menger's Theorem for digraphs.) [5]

(b) Let S and T be sets. Let f and g be real valued functions defined on S and T respectively. State the definition of a *max/min formula* for determining the maximum of f and the minimum of g. Also, describe how such a formula can be used to demonstrate that the maximum of f and the minimum of g have been obtained. [4]

(c) Let N be a directed network in which each arc e has been given a positive integer weight $c(e)$ called the capacity of e. Let x and y be vertices of N.

(i) State, without proving, a max/min formula for determining the maximum value of an xy-flow in N. [2]

(ii) Let f_0 be an xy-flow in N. Explain what it means for a path P in N to be f_0-*unsaturated.* [2]

(d) In the network below, the capacity of each arc is given as a number not in brackets, and an xy-flow, f_0, is indicated by the numbers in brackets.

Using an appropriate algorithm, find an xy-flow of maximum value in N. (You should draw an unsaturated maximal tree at each stage of your algorithm.) Also, finish by using the result you stated in part (c)(i) above, to verify that the final flow you obtain, does indeed have maximum value. [12]

Question 4

(a) Let G be a bipartite graph with bipartition $V(G) = X \cup Y$.

(i) Let $M \subseteq E(G)$. Explain what it means to say that M is a *matching*, that M is a *maximum matching* and that M is a *perfect matching.* [3]

(ii) State, without proving, a theorem of Hall that gives a necessary and sufficient condition for G to have a matching that saturates X. [2]

(b) Let G be the bipartite graph with the following adjacency table.

(In the table a '1' in row x_i and column y_j means there is an edge in G joining vertex $x_i \in X$ to vertex $y_i \in Y$ and a zero means there is no edge joining x_i and y_j.)

	y_1	y_2	y_3	y_4	y_5	y_6
x_1	1	1	0	0	1	0
x_2	0	1	0	1	1	1
x_3	0	1	0	1	0	0
x_4	0	1	1	0	0	1
x_5	0	0	0	1	1	1
x_6	0	0	0	1	0	0

Use a tree growing algorithm to find a maximum matching in G starting with the matching $M_1 = \{x_1y_2, x_3y_4, x_4y_6, x_5y_5\}$. Give a brief description of each step you take. [12]

(c) Let G be a connected bipartite graph with bipartition $V(G) = X \cup Y$ and such that $d_G(x) \geq d_G(y)$ for all pairs of vertices with $x \in X$ and $y \in Y$. Using Hall's theorem or otherwise, show that G has a matching that saturates X. [8]

Question 5

(a) Let N be a connected network. Let $\mathcal{P}$ be a property that may or may not hold for N. Explain what it means for a result to be a *good characterization* of when $\mathcal{P}$ holds for N. [2]

(b) State, without proving, a result that gives a good characterization of the property that a graph G has an Euler tour. [2]

(c) Let G be a connected graph. Prove that G has a closed walk that uses every edge of G exactly twice. [3]

(d) Let N be the network given below.

Let W_1, W_2, W_3 be the shortest walks in N which traverse every edge at least once and are such that

(i) W_1 starts and ends at v_1.

(ii) W_2 starts at v_1 and ends at v_{10}.

(iii) W_3 starts at v_1 and ends at v_7.

Suppose that $l(W_i) = w(N) + m_i$ for $i = 1, 2, 3$. Determine m_1, m_2, and m_3 giving a brief explanation of how you obtain each answer. [12]

(e) (i) Give necessary and sufficient conditions for a digraph to have a directed Euler tour. [2]

(ii) Explain why a digraph with a directed Euler tour must be strongly connected. [2]

(iii) Is it always true that a strongly connected digraph will have a directed Euler tour? Give a proof or a counterexample. [2]