UNIVERSITY OF LONDON

B. Sc. Examination 2003

MATHEMATICS

M321 Topology

Duration: 2 hours 15 minutes

Date and time:

There are FIVE questions on this paper.

Full marks may be obtained for complete answers to FOUR questions. Only your highest-scoring four questions will contribute to your total mark.

Begin each question on a new page.

Electronic calculators may be used. The make and model should be specified on the script. The calculator must not be programmed prior to the examination. Calculators which display graphics, text or algebraic equations are not allowed.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Question 1.

(a): Let $S \subseteq \mathbb{R}^n$. What does it mean to say that S is connected? [4]

Determine whether the following subsets of $\mathbb{R}$ are connected.

(i): $\mathbb{Z}$

(ii): $\{x : a_3x^3 + a_2x^2 + a_1x + a_0 = 0, \text{ for fixed constants } a_0, a_1, a_2, a_3\}$

Justify your answers. [8]

(b): Let $S \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}^n$, and let $f : S \to T$ be a function. What does it mean to say that f is continuous? [1]

Suppose that S is connected and that f is continuous and onto. Show that T is connected. [6]

(c): Let $f : [0,1] \to [0,1]$ be a continuous function. Show that there is some $t \in [0,1]$ such that $f(t) = t$. (Hint: consider the function g given by $g(x) = \frac{f(x)-x}{|f(x)-x|}$.) [6]

Question 2.

(a): Let $A \subseteq \mathbb{R}^n$. What does it mean to say that A is sequentially compact? [3]

Determine from your definition whether the following subsets of $\mathbb{R}$ are sequentially compact.

(i): $\mathbb{Z}$
(ii): $(0, 3]$
(iii): $[0, 1]$

Justify your answers. [14]

(b): Let A be a sequentially compact subset of $\mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ a continuous function. Let $B = f(A)$. Show that B is sequentially compact. [6]

State the Heine-Borel Theorem (which characterizes the sequentially compact subsets of $\mathbb{R}^n$). [2]

Question 3.

(a): Given a metric space (X, d), describe how to obtain its associated topological space. [2]

What does it mean to say that a topological space is Hausdorff? [2]

Show that the topological space associated with a metric space is Hausdorff. [4]

(b): What does it mean to say that two metrics on a set X are equivalent? [2]

Consider the metric space $(\mathbb{R}^2, d_M)$ given by

$$d_M((x_1, x_2), (y_1, y_2)) = |x_2 - x_1| + |y_2 - y_1| .$$

Draw the open ball of radius 1 with respect to d_M around the point (0,0). [2]

Show that d_M is equivalent to the usual (Euclidean) metric d_E on $\mathbb{R}^2$. [6]

Let d_B be the metric on $\mathbb{R}^2$ given by

$$d_B(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise.} \end{cases}$$

Describe the open ball around $(0, 0)$ with respect to d_B of radius $\frac{1}{2}$. Describe the open ball around $(0, 0)$ with respect to d_B of radius 2. [4]

Show that d_B is not equivalent to d_E. [3]

Question 4.

(a): Define a topological space. [3]

Let (X, τ_X) be a topolgical space and let $Y \subseteq X$. What is meant by a relatively open subset of Y? [1]

Describe the subspace topology $\tau_{X|Y}$ on Y. [1]

Prove that $(Y, \tau_{X|Y})$ is a topological space. [6]

(b): What does it mean to say that a topological space is compact? [2]

Let $X = \{a, b, c, d\}$ and $\tau_X = \{\Phi, \{a\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}$, where Φ denotes the empty set.

Show that (X, τ_X) is compact. You may *not* use (without proof) the fact that any finite topological space is compact. [3]

Is (X, τ_X) connected? Justify your answer. [3]

Find the interior and closure of each of the sets $\{b\}$ and $\{a, d\}$. [6]

Question 5.

(a): What is meant by a surface? [2]

What is a planar diagram for a surface? [3]

Draw the planar diagram representing a torus T^2, and give an associated word. [3]

Draw the planar diagram representing a projective plane P^2, and give an associated word. [3]

(b): Let $XeYeZ$ be a word representing a compact surface S, where X, Y and Z each denote a sequence of edges and e denotes a single edge. Using planar diagrams, show that S is also represented by $ffY^{-1}ZX$, for a single edge f. [4]

State, but do not prove, a theorem which classifies compact surfaces up to topological equivalence. [4]

Classify the surfaces given by the following words:

(i): $abc^{-1}d^{-1}ee^{-1}dcb^{-1}a^{-1}$,
(ii): $aba^{-1}c^{-1}dcbd$. [6]