UNIVERSITY OF LONDON

GOLDSMITHS COLLEGE

B. Sc. Examination 2003

MATHEMATICS

MT53004A (M314) Fluid Dynamics

Duration: 2 hours 15 minutes

Date and time:

There are øve questions on this paper.

Do not attempt more than FOUR questions on this paper.

Full marks will be awarded for complete answers to FOUR questions.

Electronic calculators may be used. The make and model should be speciøed on the script and the calculator must not be programmed prior to the examination.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Question 1 (a) Derive Euler's equation of motion for an inviscid Æuid in the form

$$\frac{D\mathbf{q}}{Dt} = \mathbf{F} - \frac{1}{\rho}\,\mathrm{grad}\,P,$$

where ρ is the density, $\mathbf{q}$ the velocity of the Æuid, $\mathbf{F}$ the body force per unit mass, P is the Æuid pressure, and

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\mathbf{q}.\nabla)$$

is the dioeerential operator following the Æuid. [7]

(b) At time t a Æuid Æow has the following velocity components in the x, y and z directions

$$u = 1,\ v = \frac{1}{1+5t},\ \ w = 0.$$

Find the streamline at $t = 0$ passing through the point $(0, 0, 0)$. [4]

Determine and sketch the path of the particle of Æuid which is at the point $(0, 0, 0)$ when $t = 0$, adding to the sketch the streamline at that point. [5]

(c) An incompressible, inviscid Æuid is in steady rotation under gravity with velocity components

$$u = \frac{1}{2}\Omega y,\ v = -\frac{1}{2}\Omega x,\ w = 0,$$

referred to Cartesian axes with the z-axis vertically upwards. Given that Ω is a positive constant, ρ, the density of the Æuid and g the acceleration due to gravity, show that the pressure at any point in the Æuid is given by

$$P = \frac{1}{8}\rho\Omega^2\left(x^2 + y^2\right) - \rho g z + \text{constant}.$$

[9]

Question 2 (a) Starting from Euler's equation of motion, as stated in Question 1, show that for a steady irrotational Æow of an incompressible inviscid Æuid, Bernoulli's equation takes the form

$$\frac{P}{\rho} + \frac{1}{2}\mathbf{q}^2 + gz = \text{constant},$$

where P is the pressure, ρ the density and $\mathbf{q}$ the velocity of the Æuid, z is the height above some øxed horizontal plane and g the magnitude of the acceleration due to gravity.

[8]

(b) A large cylindrical tank of radius $5a$ has a hole of radius a at its bottom. The tank at time t is ølled with water to a depth h and the speed of the water at the surface is u, while water Æows through the hole.

(i) Given that the surface and the hole are open to the atmosphere, and that the Æow is steady and irrotational throughout, show that

$$u = \frac{1}{2}\sqrt{\frac{gh}{78}}.$$

[7]

(ii) Show that h, the depth of water in the tank at time t, is given by

$$h = \left(h_0^{\frac{1}{2}} - \frac{1}{4}\sqrt{\frac{g}{78}}\,t\right)^2,$$

where h_0 is the initial depth of the water. [6]

(iii) Show that the time taken before the tank is only one quarter full is

$$t = 2\sqrt{\frac{78h_0}{g}}.$$

[4]

Question 3 (a) An incompressible, inviscid Æuid of density ρ Æows irrotationally. Show that there exists a velocity potential function, Φ, for this Æow which satisøes Laplace's equation $\nabla^2\Phi = 0$. [5]

(b) A sphere is immersed in an inønite non-viscous, incompressible Æuid of density ρ. The sphere contracts such that its centre maintains a øxed position and at time, t, its radius is R. During the contraction the Æuid moves inwards in a spherically symmetric manner, under no body forces, but remains at rest at inønity under a constant pressure P_∞.

(i) Show that the velocity potential of the Æuid at a distance r from the centre of the sphere is given by

$$\phi = \frac{R^2\dot{R}}{r} \qquad \text{for } r \geq R,$$

where $\dot{R}$ is the velocity of the surface of the sphere. [7]

(ii) Show that the pressure on the surface of the sphere is

$$P_s = P_\infty + \rho\left(R\ddot{R} + \frac{3}{2}\dot{R}^2\right).$$

[7]

(iii) Given that

$$R = A - Bt^3,$$

where A and B are constants, show that P_s has a stationary point when

$$t = \left(\frac{A}{13B}\right)^{\frac{1}{3}}.$$

[6]

(You may assume that

$$\operatorname{div}\boldsymbol{q} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(u_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}$$

in spherical polar coordinates (r, θ, ϕ), where $\boldsymbol{q}=(u_r, u_\theta, u_\phi)$ is the Æuid velocity.)

Question 4 (a) In spherical polar co-ordinates (r, θ, ϕ) the velocity potential function representing the Æuid Æow of an inviscid, incompressible Æuid past a øxed sphere of radius a centred at the origin is given by

$$\Phi = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta$$

where U is the velocity at inønity parallel to the line $\theta = \phi = 0$.

(i) Find the position of the maximum and minimum velocity on the surface of the sphere. [8]

(ii) Find the pressure at any point in the Æuid and hence show that on the surface of the sphere the pressure, P, is given by

$$P = P_\infty + \frac{1}{8}\rho U^2\left(9\cos^2\theta - 5\right),$$

where P_∞ is the pressure at inønity. [9]

(iii) Show also that the thrust exerted by the Æuid Æow on the leading hemisphere of the sphere is

$$\pi a^2\left(P_\infty - \frac{1}{16}\rho U^2\right).$$

[8]

Question 5 (a) (i) Show that for a two-dimensional motion of an incompressible, irrotational Æuid the velocity potential ϕ and the stream function ψ are related by the Cauchy-Riemann equations

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

[8]

(ii) Show also that

$$\nabla \phi . \nabla \psi = 0.$$

What can be deduced about the surfaces of constant ϕ and ψ? [5]

(b) Consider two-dimensional Æuid Æow as in (a), around a cylinder of radius a centred at the origin. If its velocity parallel to the real axis is U then its complex potential function in the region close to the cylinder is given by

$$\omega = U\left(z + \frac{a^2}{z}\right) + i\frac{k}{2\pi}\log z,$$

where $z = x + iy$.

(i) Show that the velocity components of the Æuid in the x and y directions at any point on the cylinder are

$$u = -U\left(1 - \cos 2\theta\right) - \frac{k}{2\pi a}\sin\theta, \qquad v = U\sin 2\theta + \frac{k}{2\pi a}\cos\theta,$$

where θ is the polar angle of the point. [7]

(ii) Given that $k = 4\pi a U$, show that there is only one stagnation point and that it is on the cylinder. [5]

END OF EXAMINATION