UNIVERSITY OF LONDON

B. Sc. Examination 2003

MATHEMATICS

M204 Linear Algebra

Duration: 2 hours 15 minutes

Date and time:

There are FIVE questions on this paper.

Full marks may be obtained for complete answers to FOUR questions. Only your highest-scoring four questions will contribute to your total mark.

Begin each question on a new page.

Electronic calculators may be used. The make and model should be specified on the script. The calculator must not be programmed prior to the examination. Calculators which display graphics, text or algebraic equations are not allowed.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Question 1.

(a): Let U be a subspace of a vector space V. What is meant by a basis for U? [3]

(b): Let $V = \mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$. Are the following subsets of V subspaces? Justify your answers.

$$\begin{aligned} U_1 &= \{(x, y, z) : x + 2y + 3z = 0\}, \\ U_2 &= \{(x, y, z) : x + 3y^2 = 4\}. \end{aligned}$$

[8]

If U_1 is a subspace then find a basis for U_1 and state the dimension of U_1. If U_2 is a subspace then find a basis for U_2 and state the dimension of U_2. [5]

(c): Let U and W be subspaces of some vector space V. Define the subspaces $U \cap W$ and $U + W$. [2]

State the relationship between the dimensions of U, W, $U \cap W$ and $U + W$. [2]

(d): Let P_∞ be the vector space of all real polynomials over $\mathbb{R}$. That is,

$$P_\infty = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k : a_i \in \mathbb{R}, k \geq 0\}.$$

Show that P_∞ is infinite-dimensional. (Hint: any finite subset of P_∞ has an element of maximal degree.) [5]

Question 2.

(a): Let U and V be vector spaces over a field F. Let $T : U \to V$ be a linear map.

Define what it means for T to be a linear map. [2]

Define $\mathrm{Ker}(T)$, the kernel of T, and $\mathrm{Im}(T)$, the image of T. [2]

Show that $\mathrm{Ker}(T)$ is a subspace of U. [3]

(b): Let U, V, F and T be as in part (a). Suppose that $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}\}$ is a basis for $\mathrm{Ker}(T)$ and that $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}, \mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_s}\}$ is an extension of this basis to a basis for U.

Show that the vectors $T(\mathbf{w_1}), T(\mathbf{w_2}), \ldots, T(\mathbf{w_s})$ form a basis for $\mathrm{Im}(T)$. [8]

Hence derive a relationship between the dimensions of U, $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$. [2]

(c): Let P_2 be the set of polynomials of degree at most 2 over the real numbers. That is,

$$P_2 = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}.$$

Show that the map $T : P_2 \to P_2$ given by

$$T : ax^2 + bx + c \mapsto \frac{d}{dx}(ax^2 + bx + c)$$

is linear. [3]

Let $\mathcal{B}$ be the basis $\{x^2, x, 1\}$ of P_2. Find the 3×3 matrix $T_{\mathcal{B}}$ representing T with respect to $\mathcal{B}$. [3]

Find $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$. [2]

Question 3.

(a): Let V be a vector space over a field F and let $T : V \to V$ be a linear map. Define the terms eigenvalue, eigenvector and eigenspace of T. [3]

(b): Let

$$A = \begin{pmatrix} -2 & 6 \\ -2 & 5 \end{pmatrix},$$

where the entries are considered as real numbers.

Find the eigenvalues of A and for each eigenvalue find a corresponding eigenvector. [4]

Explain why A is diagonalisable. [1]

Write down an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. [3]

Hence, or otherwise, find A^{20} (you may leave powers of integers expressed as powers of integers). [5]

(c): Consider the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ (where vectors are written as column vectors) given by

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

What is the geometric interpretation of this map? [3]

Does this map possess any eigenvectors? Justify your answer with regard to your geometric interpretation. [4]

Is it possible to diagonalise T? Justify your answer. [2]

Question 4.

(a): Let V be an n-dimensional Euclidean space.

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in V, how is $\mathbf{u} \cdot \mathbf{v}$ defined? [1]

Define what it means for $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}\}$ to be an orthonormal set. [2]

Given that the set $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}\}$ is orthonormal show that it is also a linearly independent set. Hence deduce that it is a basis for $\operatorname{span}(\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r})$. [4]

Let U be a subspace of V. Define the orthogonal complement $U^{\perp}$ of U. [2]

Show that $U \cap U^{\perp} = \{\mathbf{0}\}$. (You may use the fact that if $\|\mathbf{v}\| = 0$ then $\mathbf{v} = \mathbf{0}$.) [3]

Let the set $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}, \mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_s}\}$ be an extension of the basis $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}\}$ to an orthonormal basis for V. Show that the set $\{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_s}\}$ is a basis for $U^{\perp}$ and hence

$$\dim(U + U^{\perp}) = \dim(U) + \dim(U^{\perp}).$$

(You may use the fact that a vector $\mathbf{w}$ is orthogonal to each of the vectors $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}$ if and only if $\mathbf{w} \in U^{\perp}$.) [5]

(b): Consider $\mathbb{R}^4$. Let

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \mathbf{u_2} = \begin{pmatrix} 6 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ 5 \\ -4 \end{pmatrix}.$$

Let $U = \operatorname{span}(\mathbf{u_1}, \mathbf{u_2})$. By using the Gram-Schmidt Procedure, or otherwise, find an orthonormal basis for U. [3]

Calculate $P_U(\mathbf{v})$, the orthogonal projection of $\mathbf{v}$ onto U. [3]

Hence find the least distance between the point $\mathbf{v}$ and the subspace U. [2]

Question 5.

(a): Let V be a Euclidean space. What does it mean to say that a linear map $T : V \to V$ is orthogonal? [1]

Prove that a linear map $T : V \to V$ is orthogonal if and only if $\|T(\mathbf{v})\| = \|\mathbf{v}\|$. (Hint: consider a vector of the form $\mathbf{v_1} + \mathbf{v_2}$.) [5]

(b): What does it mean to say that an $n \times n$ matrix is symmetric? [1]

Let A be a real symmetric $n \times n$ matrix. Prove that the eigenvalues of A are real. [5]

(c): Let $Q : \mathbb{R}^2 \to \mathbb{R}$ be the quadratic form defined by

$$Q(x, y) = 2x^2 - 2xy + 2y^2$$

Find a symmetric matrix A such that $Q(\mathbf{v}) = \mathbf{v}A\mathbf{v}^T$ for all $\mathbf{v} \in \mathbb{R}^2$. [1]

Find an orthogonal matrix P and a diagonal matrix D such that $P^T AP = D$. [6]

Determine the maximum and minimum values taken by Q subject to the constraint $x^2 + y^2 = 1$. Justify your answer. [4]

Find all $(x, y) \in \mathbb{R}^2$, subject to the constraint $x^2 + y^2 = 1$, which maximise Q. [2]