UNIVERSITY OF LONDON

GOLDSMITHS COLLEGE

B. Sc. Examination 2003

MATHEMATICS

IS53015A(CIS330) Mathematical Modelling in Management Science

Duration: 3hours

Date and time:

This examination paper has two sections.
Questions 1,2,3,4 form SECTION A and questions 5,6,7,8 form SECTION B.

You should attempt THREE questions from Section A and THREE questions from Section B.
Full marks will be awarded for complete answers to six questions.

Electronic calculators may be used. The make and model should be specified on the script. The calculator must not be programmed prior to the examination. Calculators which display graphics, text or algebraic equations are not allowed.

Candidates may use GRAPH PAPER, which will be provided.

BEGIN EACH QUESTION ON A NEW PAGE and number the question and parts.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

SECTION A
You should attempt no more than three questions from this section.

Question 1

Manufacturing company NewThing have designed a combination television-kettle that boils automatically whenever the adverts come on. They have called it the AdBoil and are planning a sales strategy. They believe that demand is a function of price and estimate that if the price is £x, then the monthly demand will be $7000e^{-0.005x}$ units.

At present, each unit they sell generates a cost of £182 and they have fixed costs of £24,000 per month. They build a spreadsheet model (as shown in cells A1:E7 of the attached spreadsheet) in such a way as to automatically calculate the demand and the profit, even if the price, cost-rate per unit and fixed monthly costs change.

NOTE: There ARE values missing from the spreadsheet. You have to calculate some of them in this question. You will not calculate the values for cells B4, B5, B7 and E4.

(a) State the Excel formulae used to calculate the values in cells B4, B5 and E4. [3]

(b) Over the course of their first year selling the AdBoil, NewThing vary the price and monitor the demand as a result. They record one year's worth of figures and then use Excel to fit an Exponential curve through the data. This is shown in the attached spreadsheet. Also shown are the predictions made by NewThing's own estimate for the demand function and those made by the Exponential curve suggested by Excel.

To determine the accuracy of the original estimated demand function, Average Percentage Errors have been calculated for both sets of predictions.

(i) Calculate the value missing from cell E34. [3]

(ii) Calculate the Mean Average Percentage Error for both sets of predictions (that is the values missing from cells E44 and F44) and say why the curve fitted by Excel is the better model of how demand behaves with respect to price. [5]

(c) To start their second year, NewThing decide to set the price of the AdBoil somewhere between £500 and £700. They decide to keep using the Excel exponential curve to model demand, their fixed costs are unchanged, but due to a new accountancy structure, the variable cost associated with selling each unit is now (£75) + (26 percent of the selling price).

(i) Explain why NewThing should think very carefully before continuing to use the Exponential curve generated by the data from last year. [2]

(ii) If they fix the AdBoil price at £620, using the Excel Exponential model calculate the profit they will make in the first month of their second year. Show all your working. [7]

End of Question 1.

Question 2

Consider the following precedence table for a multi-activity project. The durations given are all in days.

Activity	Depends on	Duration
A	-	2
B	-	6
C	A	3
D	A,B	5
E	D	7
F	D	6
G	C,E	4
H	G	6
I	G,F	6
J	F	3

(a) Draw an activity network to represent the project. Use as few dummies as possible and give the vertices an acyclic ordering. [8]

(b) Find early and late times for the events in the project and indicate them clearly on your activity network. State the minimum project time. [5]

(c) Write down two possible critical paths for the project as lists of activities. [2]

(d) It is discovered that activity D actually requires 8 days of work, but ativities E and F can still begin after 5 days of work on activity D. Describe how you would modify your activity network to show this information. Also, by checking appropriate early and late times, determine whether or not this new information changes the minimum project time. [5]

End of Question 2.

Question 3

(a) A communications engineer is designing an information network that consists of six nodes $v_1, v_2, \ldots, v_6$ and cables joining them. The cable is non-directional and costs £2 per metre. The distance between the nodes in metres is shown in the adjacency table below. The gaps indicate that it is impossible to connect two nodes directly.

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	-	14	20	-	-	-
v_2	14	-	12	15	19	-
v_3	20	12	-	34	-	42
v_4	-	15	34	-	9	16
v_5	-	19	-	9	-	20
v_6	-	-	42	16	20	-

By copying out the table and running Prim's algorithm, starting with vertex v_1, determine the minimum length of cable required to connect all the nodes. Finish by drawing a diagram showing how to connect the nodes with this minimum length of cable. [8]

(b) Because the network is used to transmit information from v_1 to the other nodes, another engineer decides to use a directional cable. She redesigns the table so that every edge has a direction. The adjacency table for her network is below.

		TO					
		v_1	v_2	v_3	v_4	v_5	v_6
FROM	v_1	-	14	20	-	-	-
	v_2	-	-	12	15	19	-
	v_3	-	-	-	34	-	42
	v_4	-	-	-	-	9	16
	v_5	-	-	-	-	-	20
	v_6	-	-	-	-	-	-

(i) Draw the network represented by this table. [4]

(ii) Use the shortest path algorithm to grow a tree that shows the shortest path from v_1 to each of the other nodes. Finish by drawing the tree you generate. [8]

End of Question 3.

Question 4

A local newspaper is starting a regular feature that reviews recent rental film releases. In order to do this they wish to purchase a home cinema system for the journalists to use to view the movies. They already have a large television. They need to buy a DVD player, an amplifier and a speaker-package and they have £1000 to spend. Two local retailers each offer them a system set up that meets their budget and the newspaper must choose which to buy.

They produce the following pairwise comparison matrix to rate how important each of the three components are.

	DVD	Amp	Speaker
DVD	1	2	7
Amp	1/2	1	1/5
Speaker	1/7	5	1

However one of the reporters points out that this matrix is inconsistent. So they rethink and produce a new comparison matrix. After checking that the new matrix is consistent, they run the Analytical Hierarchy Process to determine which system to buy. The details (including the new pairwise comparison matrix) are presented on the attached spreadsheet.

NOTE: The attached spreadsheet DOES have some figures omitted. In this question you will be asked to calculate some of the missing values. We will not calculate the value missing from cell H6, but you can complete the question without it.

(a) Explain (without using an algebraic test) why the first pairwise comparison matrix is inconsistent. [4]

(b) Calculate the CI value in this situation and hence, given that the RI value for a three variable PCM is 0.58, determine whether or not the pairwise comparison matrix in cells B5:D7 is constistent, giving a reason for your answer. [5]

(c) Calculate the final scores for the machines (that is, the values that should occupy cells F23:F24). Thus state which system the newspaper should buy, giving a reason for your answer. [6]

(d) Arriving late at the meeting, one reporter looked at the weights vector in cells J5:J7 and suggested that they build their own system spending £591 on the DVD player, £334 on the amplifier and £75 on the speaker-package. Comment on two modelling issues that should be considered whilst deciding whether or not this a good idea. [5]

End of Question 4.

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SECTION B

You should attempt no more than three questions from this section.

Question 5

Consider the following linear programming problem.

Find $\mathbf{x} \in \mathbf{R}^3$ to maximize $\mathbf{z} = \mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, where

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 18 \\ 24 \\ 12 \end{pmatrix} \text{ and } \mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

(a) Prepare the problem for solution by the simplex method and construct the initial tableau. [4]

(b) Determine the first entering variable (EV) and leaving variable (LV), giving a reason for your choice in each case. [4]

(c) Complete the solution of the problem using two iterations of the simplex method. Explain carefully why your final tableau is optimal and clearly state the optimal basic feasible solution and the maximum value of z. [12]

End of Question 5.

Question 6

A food company is preparing ration packs. They have three foods, A, B and C that they can use. Each pack must contain at most 12 units of fat and at least 8 units of protein. Each kilogram of food A contains 2 units of fat and 1 units of protein. Each kilogram of food B contains 5 units of fat and 2 units of protein. Each kilogram of food C contains 6 units of fat and 3 units of protein.
Each kilogram of A costs one pound, each kilogram of B costs two pounds and each kilogram of C costs two pounds. The company want to minimize the cost of each ration pack.

(a) Show that this situation can be modelled by the following linear programming problem, P.

Find $x_1, x_2, x_3 \in \mathbf{R}$ to minimize $z = x_1 + 2x_2 + 2x_3$ subject to

$$\begin{aligned} 2x_1 + 5x_2 + 6x_3 &\leq 12 \\ 1x_1 + 2x_2 + 3x_3 &\geq 8 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

[4]

(b) Prepare P for solution by the Big M method by expressing it as a maximization problem, adding appropriate additional variables and modifying the objective function. [5]

(c) Create an initial tableau, determine the entering variable and leaving variable and complete ONE iteration of the simplex algorithm. State the basic feasible solution for the revised problem reached at the end of this iteration. [8]

(d) Explain carefully

(i) why this tableau is optimal, and

(ii) the conculsions you can draw about the original problem.

[3]

End of Question 6.

Question 7

(a) Using *either* the simplex algorithm *or* a graphical method, determine whether the following linear programming problem has an optimal solution.
Find $x_1, x_2 \in \mathbf{R}$ to maximize $z = 8x_1 + 5x_2$ subject to

$$\begin{aligned} -5x_1 + x_2 &\leq 10 \\ x_1 - 2x_2 &\leq 4 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Give brief but clear explanations of your steps and your reasoning. Any diagrams should be clearly labelled. [8]

(b) Consider the following linear programming problem.

Find $x_1, x_2 \in \mathbf{R}$ to maximize $z = x_1 + 3x_2$ subject to

$$\begin{aligned} 2x_1 + 6x_2 &\leq 27 \\ x_1 + 10x_2 &\leq 25 \\ 5x_1 + 2x_2 &\leq 58 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Slack variables x_3, x_4 and x_5 are added to the three functional constraints and The problem is tackled with the simplex method. After several iterations the following tableau is obtained.

Eqn	z	x_1	x_2	x_3	x_4	x_5	RHS
0	1	0	0	1/2	0	0	27/2
1	0	1	0	5/7	-3/7	0	60/7
2	0	0	1	-1/14	1/7	0	23/14
3	0	0	0	-24/7	13/7	1	83/7

(i) State the basic feasible solution at this point. [2]

(ii) Explain briefly how you can deduce that the problem has multiple optimal solutions. [3]

(iii) Suppose the objective function changes to $z_1 = (1+t)x_1 + 3x_2$. Determine the range of values of t for which the basic feasible solution represented by the tableau above remains optimal. [7]

End of Question 7.

Question 8

(a) Let P_1 denote the following linear programming problem.

Find $\mathbf{x} \in \mathbf{R}^n$ to maximize $\mathbf{z}_1 = \mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$,

where $\mathbf{c} \in \mathbf{R}^n, \mathbf{b} \in \mathbf{R}^m$ and A is a real $m \times n$ matrix.

(i) Formulate P_2, the **dual** problem of P_1. [3]

(ii) Let z_2 denote the objective function of P_2. Let $\mathbf{x}$ be a feasible solution to P_1 and let $\mathbf{y}$ be a feasible solution to P_2. State the *Weak Duality Theorem* for the relationship between $z_1(\mathbf{x})$ and $z_2(\mathbf{y})$. [2]

(b) We will now consider a specific problem that we shall refer to from this point as P_1.

Find $x_1, x_2 \in \mathbf{R}$ to maximize $z = 2x_1 + 3x_2$ subject to

$$\begin{aligned} x_1 + x_2 &\leq 6 \\ x_1 + 2x_2 &\leq 10 \\ 2x_1 + 5x_2 &\leq 20 \\ x_1, x_2 &\geq 0. \end{aligned}$$

(i) Formulate the dual problem, P_2. [4]

(ii) After several iterations of the simplex algorithm in the solution of P_1, the following tableau is obtained, where x_3, x_4 and x_5 are the slack variables added to the first, second and third functional constraints respectively.

Eqn	z	x_1	x_2	x_3	x_4	x_5	RHS
0	1	0	0	4/3	0	1/3	44/3
1	0	1	0	5/3	0	-1/3	10/3
2	0	0	0	-1/3	1	-1/3	4/3
3	0	0	1	-2/3	0	1/3	8/3

From this tableau find the current basic feasible solution for P_1 and the current value of z_1. [2]

(iii) Explain carefully how a feasible solution to the problem P_2 can be obtained from this tableau. Give the values of the decision variables for P_2 obtained by the method you describe. [3]

(iv) Verify that these values do indeed form a feasible solution for P_2 and calculate directly the corresponding value of z_2. [3]

(v) Are the solutions for P_1 and P_2 that you obtained from this tableau *optimal* solutions for these problems? Justify your assertions without refering to properties of the tableau above. [3]

End of Question 8.