UNIVERSITY OF LONDON

GOLDSMITHS' COLLEGE

B. Sc. Examination 2002

STATISTICS

ST53004A (ST313) Stochastic Processes

Duration: 2 hours 15 minutes

Date and time:

Answer THREE questions.
Full marks will be awarded for complete answers to THREE questions.
There are 60 marks available on this paper.
A Formula Sheet is attached to the end of this examination paper.
Electronic calculators may be used. The make and model should be specified on the script. The calculator must not be programmed prior to the examination. Calculators which display graphics, text or algebraic equations are not allowed.
NOTE: Full details of all calculations are to be shown; pre-programmed statistical tests and procedures on a calculator, apart from mean and standard deviation, must not be used.
WHITE, YEATS & SKIPWORTH: Tables for Statisticians to be provided.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Question 1 The probability generating function (p.g.f.) of a linear birth-death process $N(t)$ is given by

$$G(z,t) = \left[\frac{\mu(1-z) + (\lambda z - \mu)e^{-(\lambda-\mu)t}}{\lambda(1-z) + (\lambda z - \mu)e^{-(\lambda-\mu)t}}\right]^{n_0},$$

where $\lambda > 0$ is the birth rate per individual, $\mu > 0$ is the death rate, $N(0) = n_0$ is the initial size of the population, and $\lambda \neq \mu$.

(a) Find the mean population size at time t and interpret your result. [6]

(b) By putting $\lambda = \mu + x$ and letting $x \to 0$, show that the p.g.f. in the case $\lambda = \mu$ is given by

$$G(z,t) = \left[\frac{\mu t(1-z) + z}{\mu t(1-z) + 1}\right]^{n_0}.$$

Is the process now stationary? [5]

(c) Find the probability of extinction by time t: $p_0(t) = \Pr[N(t) = 0]$ in each of the above cases, and show that ultimate extinction is certain when $\lambda \leq \mu$, but when $\lambda > \mu$ the probability of ultimate extinction is $(\mu/\lambda)^{n_0}$. [9]

Question 2 Consider the general birth-death process with birth rates λ_n and death rates μ_n, all assumed to be positive (except $\mu_0 = 0$). Show from the forward equations that if an equilibrium (stationary) distribution exists, then

$$S = 1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n}$$

is convergent. [6]

(a) A fast food store has a single server with exponential service time mean $1/\beta$, and capacity for L queueing customers (including the one being served). Arrivals follow a Poisson process rate $\alpha > 0$. Find the equilibrium distribution, the proportion of idle time and the proportion of customers who turn away when the queue is full. [6]

(b) Find these quantities when there are k servers each with the same service rate. ($k < L$) [8]

Question 3 (a) Determine the closed equivalence classes of the finite Markov chain with transition matrix given below. Hence classify the states as transient, null or positive recurrent (persistent), periodic or aperiodic.

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 2/3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

[10]

(b) Find the stationary distribution and hence the mean recurrence times for all states in any positive recurrent (persistent) class. [5]

(c) Find the probabilities of eventual absorption, starting in the transient states, into the closed classes, and show that eventual absorption is certain. [5]

Question 4 (a) Define the first return probability $f_{ii}^{(n)}$ for state i at time n, and give an expression for the mean recurrence time $\mathrm{E}[T_{ii}]$. Give also a condition for the recurrence (persistence) of state i. [5]

(b) A model for the number of people sharing a flat (apartment) for singles postulates that at each time period the number remains at one with probability $\alpha > 0$, otherwise increases to two. Similarly with 3 people, it remains there with probability $\beta > 0$, otherwise goes back to one. Whenever there are just two the number increases to three with certainty.

(i) Draw a flow diagram marking the probabilities of all possible flows. [3]

(ii) Hence write down the transition matrix. [2]

(iii) Find the first return probabilities for state 3 directly, and hence show that this state is persistent (recurrent). Find its mean recurrence time. [8]

(iv) Write down the mean recurrence time for state 1. [2]

Question 5 (a) In the gambler's ruin problem with initial capital £k, show that the probability of ruin is

$$y_k = \frac{(q/p)^a - (q/p)^k}{(q/p)^a - 1} \qquad (p \neq q)$$
$$= 1 - \frac{k}{a} \qquad (p = q = \frac{1}{2})$$

where $p > 0$ is the probability of winning £1, $q = 1 - p > 0$ is the probability of losing £1, and £a is the retirement capital. [10]

(b) Three contestants A, B, C attempt to win a prize in a game show. A goes first, then B, then C. At each turn a contestant answers a series of questions. If he answers correctly, he gains one point. If he answers incorrectly (or does not answer) he loses a point. Play passes to the next player if his score falls below the inital score of one. The player wins the prize if his score reaches four points. If the probabilities of answering correctly (assume independence for each question) are 1/4, 1/3 and 2/5 for A, B and C respectively.

What is the probability that (i) A (ii) C (iii) one of the contestants win the prize? [5]

If play continues in the same order until one of the contestants wins the prize, what is the probability it is C? [5]