UNIVERSITY OF LONDON

GOLDSMITHS COLLEGE

B. Sc. Examination 2002

MATHEMATICS

MT53011A (M335) Optimization

Duration: 2 hours 15 minutes

Date and time:

Do not attempt more than FOUR questions on this paper. Full marks will be awarded for complete answers to FOUR questions.

Electronic calculators may be used. The make and model should be specified on the script. The calculator must not be programmed prior to the examination. Calculators which display graphics, text or algebraic equations are not allowed.

Candidates may use graph paper, which will be provided.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Question 1 Consider the following linear programming problem:
Find $\mathbf{x} \in \mathbb{R}^n$ to maximize $z = \mathbf{c}^\mathbf{T}\mathbf{x}$ subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

(a) Show that the feasible dictionary with basic variables $\mathbf{x_B}$ and non-basic variables $\mathbf{x_N}$ can be represented in the matrix form

$$\begin{aligned} \mathbf{x_B} &= \mathbf{A_B^{-1}b} - \mathbf{A_B^{-1}A_N x_N} \\ z &= \mathbf{c_B^T A_B^{-1} b} + (\mathbf{c_N^T} - \mathbf{c_B^T A_B^{-1} A_N})\mathbf{x_N} \end{aligned}$$

for suitably defined matrices $\mathbf{A_B}$ and $\mathbf{A_N}$ and vectors $\mathbf{c_B}$ and $\mathbf{c_N}$. [6]

(b) Suppose

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix},$$

$\mathbf{x^T} = (x_1, x_2, x_3, x_4, x_5)$, $\mathbf{b^T} = (150, 80)$ and $\mathbf{c^T} = (3, 4, 2, 0, 0)$. Describe the Revised Simplex Algorithm by performing ONE iteration of the algorithm starting from the basic feasible solution $x_1 = 0, x_2 = 15, x_3 = 0, x_4 = 0, x_5 = 5$. State the next basic feasible solution. [13]

(c) Determine whether the basic feasible solution which you obtained in (b) is optimal. [6]

Question 2 An oil refinery can produce three different grades of gasoline G_1, G_2, G_3 from two types of crude oil C_1, C_2. The numbers of barrels of crude oil needed to produce one barrel of each grade of gasoline, and the resulting net profit, are as follows:

	Gasoline		
	G_1	G_2	G_3
C_1 required	4	2	6
C_2 required	3	4	2
Net Profit ($/barrel)	3.2	2	4

The refinery has 90 thousand barrels of C_1 and 80 thousand barrels of C_2 in store.

The problem of determining how much of each grade of gasoline should be produced to maximise profit can be stated as the linear programming problem:
Find $x_1, x_2, x_3 \in \mathbf{R}$ to maximise $z = 3.2x_1 + 2x_2 + 4x_3$ subject to

$$\begin{aligned} 4x_1 + 2x_2 + 6x_3 &\leq 90, \\ 3x_1 + 4x_2 + 2x_3 &\leq 80, \end{aligned}$$

and $x_i \geq 0$ for $1 \leq i \leq 3$.

The optimal solution to this problem is given by the following feasible dictionary, where x_4 and x_5 are the slack variables which are added to the first and second constraints, respectively.

$$\begin{aligned} x_2 &= 5 + x_3 + 0.3x_4 - 0.4x_5 \\ x_1 &= 20 - 2x_3 - 0.4x_4 + 0.2x_5 \\ z &= 74 - 0.4x_3 - 0.68x_4 - 0.16x_5 \end{aligned}$$

Solve the following independent variations of the problem.

(a) The refinery can buy extra barrels of C_2 at a cost of \$0.06/barrel. How much should it buy and what is the new optimal solution and maximum profit? [10]

(b) The profit for G_3 changes from \$4/barrel to \$(4+p)/barrel. Determine the optimal solution and maximum profit for all p, with $0 \leq p \leq 2$. [15]

Question 3 Consider the following linear programming problem:
Find $x_1, x_2 \in \mathbf{R}$ to maximise $z = 3x_1 + 5x_2$ subject to

$$\begin{aligned} x_1 + \ \ x_2 &\leq 3, \\ x_1 + 3x_2 &\leq 6, \end{aligned}$$

and $x_1, x_2 \geq 0$.

The optimal solution to this problem is given by the following feasible dictionary, where x_3 and x_4 are the slack variables which are added to the first and second constraints, respectively.

$$\begin{aligned} x_1 &= \tfrac{3}{2} + \tfrac{1}{2}x_4 - \tfrac{3}{2}x_3 \\ x_2 &= \tfrac{3}{2} - \tfrac{1}{2}x_4 + \tfrac{1}{2}x_3 \\ z &= 12 - x_4 - 2x_3 \end{aligned}$$

(a) Use the branch and bound algorithm to find an optimal integer solution to the above problem, choosing x_1 as your first branching variable. [22]

(b) Draw a branch and bound tree to illustrate your solution. [3]

Question 4 The owner of a chain of three grocery stores has purchased five crates of bananas. She wishes to know how she should allocate the five crates to the three stores in order to maximise her expected profit. (She can only allocate whole numbers of crates to each store). The following table gives the expected profit at each store when it is allocated various numbers of crates.

	Stores		
Number of crates	S_1	S_2	S_3
0	0	0	0
1	6	3	3
2	11	4	4
3	12	5	6
4	12	6	9
5	12	7	10

Use dynamic programming to find ALL the optimal solutions to this problem. [18]
Take care to define:

(a) the decision variables and state variables; [2]

(b) the optimisation problem which you solve at stage i and with state d_i; [2]

(c) the recurrence relation which you use to obtain your solutions; [3]

Question 5 Use the method of Lagrange multipliers to solve the following optimization problem.

Find $x_1, x_2 \in \mathbf{R}$ to maximise $z = 2x_1x_2 - x_1^2 - 2x_2$ subject to $x_1 + x_2 \leq 6$, $x_1 \geq 0$, and $x_2 \geq 0$.

Give brief descriptions of the steps in your solution. [25]