UNIVERSITY OF LONDON

GOLDSMITHS COLLEGE

B. Sc. Examination 2002

MATHEMATICS

MT53007A (M331) Graph Theory

Duration: 2 hours 15 minutes

Date and time:

Do not attempt more than FOUR questions on this paper. Full marks will be awarded for complete answers to FOUR questions.

Electronic calculators may be used. The make and model should be specified on the script. The calculator must not be programmed prior to the examination. Calculators which display graphics, text or algebraic equations are not allowed.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Question 1 Let G be a graph.

(a) Explain what it means to say that a subgraph T of G is a *tree*. [2]

(b) Describe a recursive algorithm for constructing a maximal tree which contains a given vertex v_1 in G. [3]

(c) Suppose that G is connected and that the edges of G have been given integer weights. Explain how the algorithm you described in (b) can be modified to construct:

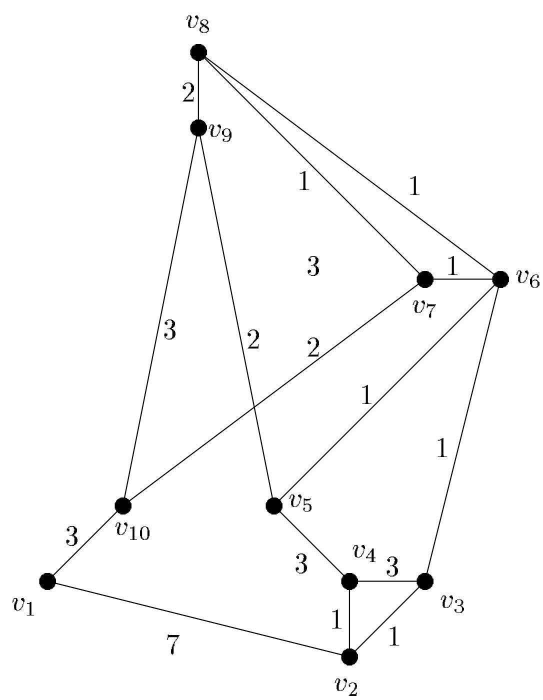
(i) a breadth first search spanning tree of G rooted at v_1;

(ii) a depth first search spanning tree of G rooted at v_1;

(iii) a maximum weight spanning tree of G.

[3]

(d) Use the algorithms you described in (c) to construct a depth first search spanning tree rooted at v_1 and a maximum weight spanning tree of the following network. (For each tree, label the vertices as $x_1 = v_1$, $x_2 = v_{10}$, etc., to show the order in which you choose them using the recursive algorithm.)



[6]

(e) Let D be a digraph and H be a subgraph of D.

(i) Explain what it means to say that H is *strongly connected* and that H is a *strongly connected component* of D. [2]

(ii) Explain how the algorithm you described in (b) can be modified to construct the strongly connected components of D. [3]

(iii) Use the modified algorithm to construct the strongly connected components of the digraph D_0 with the following adjacency table. (In this adjacency table a '1' in row v_i and column v_j means there is an arc in D_0 from vertex v_i to vertex v_j and a '0' means there is no arc in D_0 from v_i to v_j.)

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	0	1	1	1	1	0	1
v_2	0	0	1	0	0	1	1
v_3	0	0	0	0	0	0	1
v_4	1	1	1	0	1	1	1
v_5	0	1	1	0	0	0	1
v_6	0	0	1	0	1	0	1
v_7	0	0	1	0	0	0	0

[6]

Question 2 (a) Let S and T be sets. Let f and g be real valued functions defined on S and T, respectively. Explain what it means for an equation to be a *max/min formula* for f and g. Explain how such a formula is useful for solving optimization problems. [3]

(b) Let N be a directed network in which each arc e has been given a non-negative weight $c(e)$ called the capacity of e. Let x and y be vertices of N.

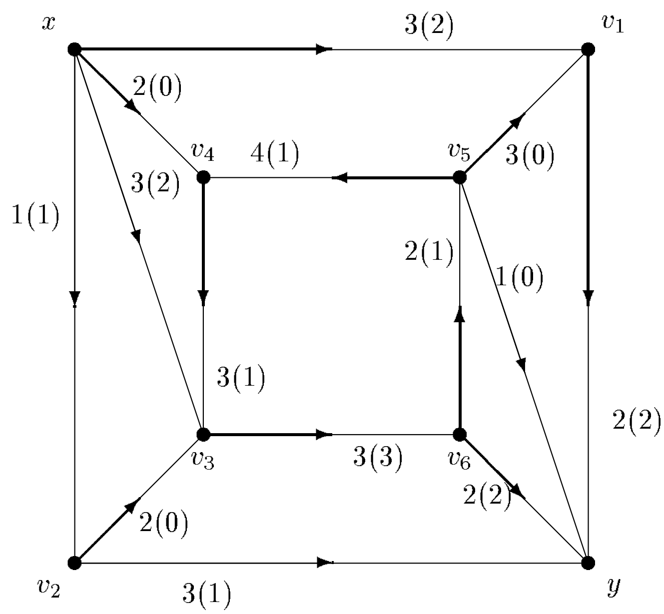
(i) Explain what it means for a real valued function f defined on $A(N)$ to be an xy-flow in N. Define the value of the flow. [3]

(ii) Explain what it means for a set of arcs of N to be an xy-cut in N. Define the capacity of the cut. [2]

(iii) State, without proving, a max/min formula for determining the maximum value of an xy-flow in N. [2]

(c) In the network N_1, shown below, a flow f_0 is indicated by the numbers in brackets and the capacity of each arc by the numbers not in brackets.

$N_1 =$



Use the maximum flow algorithm to find a flow of maximum value from x to y in N_1, starting with the given flow f_0, and giving a brief description of the steps in the algorithm.
Use your result from (b)(iii) to demonstrate that the flow you obtain does indeed have maximum value. [10]

(d) Let D_1 be the digraph obtained by ignoring the capacities and flows on the arcs of the network N_1 given in (c). Determine a maximum set of pairwise arc-disjoint directed xy-paths in D_1. Justify the fact that the set you obtain is indeed maximum. [5]

Question 3 (a) Explain what it means to say that a set of edges M in a graph G is a *matching* and to say that a path P in G is *M-augmenting.* [3]

(b) Let G be a bipartite graph with bipartition X, Y. State the result of Hall which gives a necessary and sufficient condition for G to have a matching which saturates X. [2]

(c) Use augmenting paths to determine whether the bipartite graph with the following bipartite adjacency matrix has a perfect matching, starting from the matching $M_1 = \{x_1y_1, x_2y_2, x_3y_3, x_4y_5, x_5y_4, x_6y_8, x_8y_6\}$. Give a brief description of the steps in your algorithm.

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9
x_1	1	1	0	1	0	1	1	0	1
x_2	1	1	0	0	1	0	0	1	0
x_3	0	1	1	0	1	1	1	0	1
x_4	1	0	0	0	1	0	0	1	0
x_5	1	0	1	1	1	1	0	1	0
x_6	0	1	0	0	1	0	0	1	0
x_7	1	1	0	0	0	0	0	1	0
x_8	0	1	1	0	1	1	1	1	1
x_9	1	1	0	0	1	0	0	1	0

Use Hall's theorem to justify your answer. [10]

(d) State the result of Tutte which gives a necessary and sufficient condition for a graph G to have a perfect matching. [3]

(e) Construct a 2-edge-connected graph G such that G has an even number of vertices, each vertex of G has degree four and G has no perfect matching. Use Tutte's theorem to demonstrate that the graph you construct has no perfect matching. [7]

Question 4 Let R be a network obtained by giving each edge e of the complete bipartite graph $K_{m,m}$ an integer weight $w(e)$.

(a) Explain what it means for a function l from $V(R)$ into the integers to be a *feasible vertex labelling*. Define the *equality subgraph* $R(l)$. [4]

(b) State, without proving, a max/min formula which determines the maximum weight of a perfect matching in R. [2]

(c) Describe an algorithm which finds a perfect matching of maximum weight in R. [8]

(d) Use the algorithm for the optimal assignment problem to find a maximum weight perfect matching in the weighted complete bipartite graph $K_{5,5}$ with weights given by the following matrix.

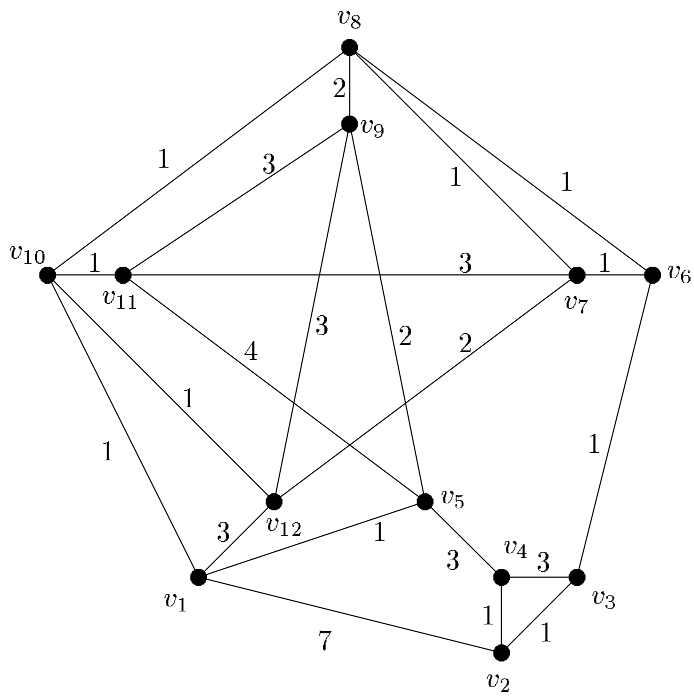
	y_1	y_2	y_3	y_4	y_5
x_1	5	1	2	3	3
x_2	4	3	4	4	3
x_3	3	2	5	6	2
x_4	1	2	3	2	1
x_5	1	2	1	2	1

Use the max/min formula you stated in (ii) to verify that the matching you have obtained does indeed have maximum weight. [11]

Question 5 (a) State, without proving, necessary and sufficient conditions for a graph G to have an Euler tour. [2]

(b) Explain how the problem of finding a shortest closed walk which traverses every edge of a connected network N can be converted to the problem of finding a minimum weight perfect matching in a complete graph. [8]

(c) Let N be the network shown below.



(i) Use Dijkstra's algorithm to construct a spanning tree which contains shortest paths from v_6 to every vertex of N. [7]

(ii) Let W_1 and W_2 be the shortest walks in N which traverse every arc of N at least once, and are such that:

- W_1 starts and ends at v_6;
- W_2 starts at v_2 and ends at v_3.

Suppose $\ell(W_i) = w(N) + m_i$ for $i \in \{1, 2\}$. Determine m_1 and m_2 giving a short explanation of how you obtain each answer. [8]