UNIVERSITY OF LONDON

GOLDSMITHS COLLEGE

B. Sc. Examination 2002

MATHEMATICS

MT53004A(M314) Fluid Dynamics

Duration: 2 hours 15 minutes

Date and time:

Do not attempt more than FOUR questions on this paper.
Full marks will be awarded for complete answers to FOUR questions.
Electronic calculators may be used. The make and model should be specified on the script and the calculator must not be programmed prior to the examination.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Question 1 (a) Derive Euler's equation of motion for an inviscid fluid in the form

$$\frac{D\mathbf{q}}{Dt} = -\frac{1}{\rho}\,\text{grad}\,P - \text{grad}V,$$

where ρ is the density, $\mathbf{q}$ the velocity of the fluid, $-\text{grad}V$ is the gradient of the potential energy V due to the body force acting, P is the fluid pressure, and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{q}.\nabla)$$

is the differential operator following the fluid. [7]

(b) At any time t a fluid flow has the following velocity components in the x, y and z directions

$$u = 1,\ v = 1 + 6t,\ w = 0.$$

Find the streamline at $t = 0$ passing through the point $(0, 0, 0)$. [4]

Determine and sketch the path of the particle of fluid which is at the point $(0, 0, 0)$ when $t = 0$, adding to the sketch the streamline at that point. [5]

(c) An incompressible, inviscid fluid is in steady rotation under gravity with velocity components

$$u_r = 0,\ u_\theta = \omega r,\ u_z = 0,$$

referred to cylindrical polar axes with the z-axis vertically upwards. Given that ω is a constant, show that the pressure at any point in the fluid is given by

$$P = \frac{1}{2}\rho\omega^2 r^2 - \rho g z + constant,$$

where $r^2 = x^2 + y^2$ and g is the gravitational acceleration.

[9]

Question 2 (a) Starting from Euler's equation of motion, stated in question 1, show that for a steady irrotational flow of an incompressible inviscid fluid, Bernoulli's equation takes the form

$$\frac{P}{\rho} + \frac{1}{2}\mathbf{q}^2 + V = constant,$$

where P is the pressure, ρ the density, $\mathbf{q}$ the velocity of the fluid and V is the potential due to conservative body forces acting on the fluid.

[8]

(b) An incompressible inviscid liquid is contained between two co-axial cylinders of inner and outer radii a and b respectively, the common axis is vertical and coincides with the z-axis. The upper surface of the liquid is open to the atmosphere at pressure P_A and the liquid is bounded below by a rigid plane base $z = 0$. The liquid is in motion under gravity so that the rectangular components of velocity at the point (x, y, z) are

$$\left(-\frac{\omega a^2 y}{2r^2}, \frac{\omega a^2 x}{2r^2}, 0\right),$$

where $r^2 = x^2 + y^2$ and ω is a constant.

(i) Verify that the motion is irrotational. [5]

(ii) Show that when the depth is a at the inner boundary and $2a$ at the outer boundary, the value of ω is given by

$$\omega^2 = \frac{8gb^2}{a\left(b^2 - a^2\right)},$$

where g is the acceleration due to gravity. [12]

Question 3 (a) In a fluid containing no source or sink, show that the density ρ and velocity $\boldsymbol{q}$ of the fluid satisfy the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{q}) = 0.$$

[8]

(b) A sphere, whose centre is fixed and has a radius, R, at time t given by

$$R = B + At^2,$$

where A and B are constants, is immersed in an infinite non-viscous, incompressible fluid of density ρ. As the sphere contracts the fluid moves inwards in a spherically symmetric manner, under no body forces, but remains at rest at infinity under a constant pressure P_∞. Show that the velocity potential of the fluid at a distance r from the centre of the sphere is given by

$$\phi = \frac{f(t)}{r} \qquad \text{for } r \geq R,$$

where $f(t)$ is an arbitrary function of time. [7]

(c) Show that the pressure on the surface of the sphere is

$$P_s = P_\infty + 2A\rho\left(B + 4At^2\right).$$

[10]

(You may assume that

$$\operatorname{div} \boldsymbol{q} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(u_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}$$

in spherical polar coordinates (r, θ, ϕ), where $\boldsymbol{q}=(u_r, u_\theta, u_\phi)$ is the fluid velocity.)

Question 4 (a) An incompressible, inviscid fluid of density ρ flows irrotationally. Show that there exists a velocity potential function, Φ, for this flow which satisfies Laplace's equation $\nabla^2\Phi = 0$. [5]

(b) Given that $\Phi = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta$ is the velocity potential function representing the fluid flow of an inviscid, incompressible fluid past a fixed sphere of radius a with a velocity U parallel to the line $\theta = 0$, show that the streamlines of this flow satisfy the equation

$$(r^3 - a^3)\sin^2\theta = \mathrm{C}r,$$

where C is a constant. Sketch the streamlines indicating particularly the streamline for C=0. [12]