UNIVERSITY OF LONDON

GOLDSMITHS COLLEGE

B. Sc. Examination 2002

MATHEMATICS

MT53002A (M310) Complex Analysis

Duration: 2 hours 15 minutes

Date and time:

Do not attempt more than FOUR questions on this paper.

Full marks will be awarded for complete answers to FOUR questions.

Electronic calculators may be used. The make and model should be specified on the script. The calculator must not be programmed prior to the examination. Calculators which display graphics, text or algebraic equations are not allowed.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Question 1 (a) Write $z = R(\cos\theta + i\sin\theta)$ and $w = r(\cos\phi + i\sin\phi)$. Show that

$$|z+w| = \sqrt{R^2 + 2Rr\cos(\theta-\phi) + r^2}.$$

[3]

(b) By considering the real and imaginary parts of e^z, or otherwise, show that the solutions of $e^z = 1$ are

$$z = 2n\pi i \qquad \text{for } n \in \mathbb{Z}.$$

You may assume the identity $e^{x+iy} = e^x(\cos y + i\sin y)$. [5]

(c) Define $f(z) = \bar{z}(z^2 - 1)$ for all $z \in \mathbb{C}$.

(i) Find real valued functions $u(x,y)$ and $v(x,y)$ such that

$$f(x+iy) = u(x,y) + iv(x,y)$$

for all $x + iy \in \mathbb{C}$. [4]

(ii) Show that f satisfies the Cauchy–Riemann equations at $z = x + iy$ if and only if

$$x^2 - y^2 = 1 \quad \text{and} \quad xy = 0.$$

[6]

(iii) Hence deduce that f is differentiable at -1 and 1, but nowhere else. [5]

(iv) Give the values of $f'(-1)$ and $f'(1)$ [2]

Here $\bar{z}$ *denotes the complex conjugate of* z.

Question 2 (a) Given a function

$$u(x,y) = 4xy^3 - 4x^3y,$$

find a real valued function $v(x,y)$ such that

$$f(x+iy) = u(x,y) + iv(x,y)$$

is analytic on $\mathbb{C}$. [7]

(b) Let $f(z)$ be an entire function, let $\alpha \in \mathbb{C}$ and let γ be a closed path whose image does not contain α. Write down Cauchy's formula for

$$\int_\gamma \frac{f(z)}{(z-\alpha)^{n+1}}\,\mathrm{d}z.$$

[3]

(c) Evaluate the path integral

$$\int_\gamma \frac{z\sin \pi z}{(z-1)^3}\,\mathrm{d}z,$$

where γ is the positively oriented circle centred at 0 with radius 2. [5]

(d) (i) Show that for f entire and $r > 0$,

$$f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + re^{i\theta})\,\mathrm{d}\theta.$$

[5]

(ii) Hence by considering $\alpha = 0$, $r = 1$ and $f(z) = \exp z^n$, show that

$$\int_0^{2\pi} \exp(\cos n\theta)\cos(\sin n\theta)\,\mathrm{d}\theta = 2\pi.$$

[5]

Question 3 (a) Define $\gamma\colon [0,1] \to \mathbb{C}$ by $\gamma(t) = t^2 + it^3$.

(i) Find the length of γ from first principles. [4]

(ii) Compute the path integral

$$\int_\gamma \bar{z}\,\mathrm{d}z.$$

[4]

(iii) Let δ be the straight line from $\gamma(0)$ to $\gamma(1)$. Compute the path integral

$$\int_\delta \bar{z}\,\mathrm{d}z.$$

[3]

Here $\bar{z}$ denotes the complex conjugate of z.

(b) Locate and classify the isolated singularities of

(i) $\dfrac{z - \sin z}{z^3}$.

(ii) $\dfrac{\exp z^2}{z^2}$,

(iii) $\dfrac{1}{\sin 1/z}$,

Further in the case of a pole compute the residue at the pole. [14]

You may assume the solutions of $\sin(z) = 0$ *are*

$$z = n\pi \qquad \textit{for } n \in \mathbb{Z}.$$

Question 4 (a) Let f be a nonconstant entire function with $f(\alpha) = 0$ for some $\alpha \in \mathbb{C}$.

(i) Show that there exists an integer $m \geq 1$ and an entire function g such that $f(z) = (z-\alpha)^m g(z)$ and $g(\alpha) \neq 0$. [8]

(ii) Hence deduce that α is not a limit point of the zeros of f. [5]

(b) Find the Laurent expansion of the function

$$\frac{1}{z^3 - z}$$

in the region $|z-1| > 2$. [7]

(c) Evaluate the following path integral

$$\int_\gamma \frac{\exp z^2}{(2z+1)(2z-1)}\,\mathrm{d}z$$

where γ is the positively oriented unit circle with center 1. [5]

Question 5 (a) Let f be an analytic function on a region $\Omega \subseteq \mathbb{C}$ except at a point $\alpha \in \Omega$ and let $m \geq 1$ be an integer. Show that if

$$\lim_{z \to \alpha} (z - \alpha)^m f(z)$$

exists and is not 0 then f has a pole of order m at α. [6]

You may assume that if g is an analytic function on $\Omega \setminus \{\alpha\}$ and

$$\lim_{z \to \alpha} g(z)$$

exists then g has a removable singularity at α.

(b) Evaluate the integrals

(i) $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{3 + 2\cos\theta}$; [9]

(ii) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)^2}\,\mathrm{d}x$ (where $a > 0$). [10]