**UNIVERSITY OF LONDON**

**GOLDSMITHS COLLEGE**

**B. Sc. Examination 2002**

# MATHEMATICS

## MT52016A(M227) Analysis and Applied Mathematics

**Duration: 3 hours**

**Date and time:**

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*Do not attempt more than THREE questions in section A. Do not attempt more than THREE questions in section B.*

*Full marks will be awarded for complete answers to SIX questions; three from section A and three from section B. USE SEPARATE BOOKLETS FOR SECTION A AND SECTION B.*

*Electronic calculators may be used. The make and model should be specified on the script and the calculator must not be programmed prior to the examination.*

**THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM**

## 0.1 SECTION B

**Question 1** (a) Use the $D$-operator method or otherwise to solve the following differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 4e^{-x}\sin 2x.$$

[10]

(b) Solve for $x$ and $y$, using the $D$-operator method, or otherwise, the following system of differential equations

$$\frac{dx}{dt} + 2x + y = 0$$

$$\frac{dy}{dt} + x + 2y = e^t$$

given that $x = 0, y = 0$ at $t = 0$. [10]

**Question 2** The differential equation

$$2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (1 - 2x)\,y = 0,$$

has solutions of the form $y = \sum_{r=0}^{\infty} a_r x^{c+r}$.

(a) Show that $c = \frac{1}{2}$ or 1. [6]

(b) Show also that

$$a_r = -\frac{2}{(c+r)(2c+2r-3)+1}\,a_{r-1},$$

where $r = 1, 2, 3 \ldots$. [5]

(c) Hence obtain two linearly independent solutions of the differential equation, and calculate the first four terms in each solution. [9]

**Question 3** (a) A function $f(x)$ of period $2\pi$ is defined in the interval $[-\pi, \pi]$ by

$$\begin{aligned} f(x) &= 0, \; -\pi \le x < 0 \\ &= x, \quad 0 \le x \le \pi. \end{aligned}$$

(i) Sketch the graph of $f(x)$ in the range $-3\pi < x < 3\pi$. [4]

(ii) Find the Fourier series for $f(x)$. [16]

**Question 4** Let $y(t)$ be a function such that

$$\lim_{t\to\infty} e^{-st}y(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} e^{-st}\frac{dy}{dt}(t) = 0 \text{ for a given parameter } s.$$

(a) Given that the Laplace transform of $y(t)$ is defined by

$$L\{y(t)\} = \int_0^\infty e^{-st}y(t)dt = \bar{y}(s),$$

where $\bar{y}(s)$ is a function of the parameter $s$, show that for $s > a$

$$L\{e^{at}\} = \frac{1}{s-a}.$$

[3]

(b) Show also that for $s > a$

$$L\{te^{at}\} = \frac{1}{(s-a)^2}.$$

[3]

(c) Show that

$$L\{\frac{dy}{dt}(t)\} = sL\{y(t)\} - y(0),$$

[3]

(d) Apply Laplace transforms to solve the differential equation

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 6te^{2t},$$

given that

$$y = 1 \text{ and } \frac{dy}{dt} = 1, \text{when } t = 0.$$

It may be assumed that

$$L\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}},$$

and

$$L\{\frac{d^2y}{dt^2}(t)\} = s^2L\{y(t)\} - sy(0) - \frac{dy}{dt}(0).$$

[11]

**Question 5** Find the solution of the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2},$$

where $c$ is a real constant and $y(x,t)$ is the displacement of a string of length $\ell$ which satisfies the following conditions

$$\begin{aligned} y &= 0 \text{ at } x=0 \text{ and at } x=\ell \text{ for all values of } t, \\ y &= a\sin\frac{\pi x}{\ell} \text{ when } t=0 \text{ for all } x\in[0,\ell], \\ \frac{\partial y}{\partial t} &= b\sin\frac{\pi x}{\ell} \text{ when } t=0 \text{ for all } x\in[0,\ell]. \end{aligned}$$

[20]

**Question 6** (a) Find div $\boldsymbol{A}$ and curl $\boldsymbol{A}$ given that

$$\boldsymbol{A} = x^2y\boldsymbol{i} - yz\boldsymbol{j} + 3y^2z\boldsymbol{k}.$$

[6]

(b) (i). Evaluate

$$\oint_c x^2ydx + xy^2dy,$$

where $c$ is the closed curve of the region bounded by $y=x^2$ and $y=x$, and is traversed in the positive anti-clockwise sense. [7]

(ii). Check this result by use of Green's Theorem. [7]

**END OF EXAMINATION**