UNIVERSITY OF LONDON

GOLDSMITHS COLLEGE

B. Sc. Examination 2002

MATHEMATICS

MT52015A (M230) Mathematical Modelling in Management Science

Duration: 3 hours

Date and time:

Answer FIVE questions.

Full marks will be awarded for complete answers to FIVE questions.

There are 125 marks available on this paper.

Electronic calculators may be used. The make and model should be specified on the script. The calculator must not be programmed prior to the examination. Calculators which display graphics, text or algebraic equations are not allowed.

GRAPH PAPER is required for this examination.

Begin each question on a new page and number the question and parts.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Question 1

Northdown Electronics makes components for a major manufacturer of aircraft engines. The manufacturer notifies the Northdown sales office each quarter of its requirements for the next three months. These can vary quite widely according to the type of engine that the manufacturer is producing. The table below shows the demand for two components, coded X and Y, for the coming three months.

Demand	Jul	Aug	Sep
X	1000	3000	5000
Y	1000	500	3000

Northdown has a total production capacity for X and Y taken together of 6000 units per month and sufficient storage capacity to hold a total of at most 2500 units after the demand for that month has been met (here we assume for simplicity that production takes place during the month and demand is met at the end of the month). The forecasted unit production costs in each month are given in the table below.

Production cost per unit	Jul	Aug	Sep
X	£20.00	£21.50	£19.50
Y	£10.20	£10.80	£10.00

The average holding cost per unit for each component at the end of any month is estimated at 2% of the production costs for that component during the month (this cost includes the cost of storage and money tied up in inventory). At the start of July, there will be no unit of either component in inventory. Northdown requires a production schedule which minimises the total production and holding costs for components X and Y for the months of July to September.

This problem has been modelled on the attached spreadsheet as a linear programming problem. The solution shown is optimal and has been found using the Solver tool in Excel.

(a) Describe the model used. You may use (if you wish) the range names shown in the textbox on the spreadsheet and refer to appropriate Excel functions. State carefully the objective and all constraints, including any constraints not shown on the spreadsheet. [16]

(b) Consider how the demand for each component is satisfied in each of the months July, August, September. How much of each monthly demand is met from inventory and how much is met from current production? [5]

(c) Suppose that Northdown is able to raise the maximum total production capacity in September (only) to 6500 units, while maintaining the production costs given in the table above. Can the company use this extra production capacity to reduce the total cost of meeting the given demand? Justify your answer. [4]

Question 2

Bangers is a small company which buys second hand cars and trucks to repair and refurbish for resale. Each vehicle must be processed in the engine shop, the body shop and the paint shop. The average number of hours work which each type of vehicle requires in each shop is shown in the table below.

	work-hours per car	*work-hours per truck*
Engine shop	24	15
Body shop	40	40
Paint shop	20	30

Bangers have at most 1320 work-hours available per week in the engine shop, at most 2400 work-hours per week in the body shop and at most 1500 work-hours per week in the paint shop. Each car contributes on average £1500 to profit and each truck contributes on average £2000 to profit. We assume there is an unlimited supply of cars and trucks available to the firm for refurbishment and that it can resell all the vehicles it processes. Bangers requires a weekly production schedule that will maximise its profit.

(a) Formulate the problem of determining the number of cars and the number of trucks that the firm should process each week as a linear programming problem. Define the decision variables and say briefly how the objective function and each constraint is obtained. [7]

(b) Plot the constraint lines for this linear programming problem, either using the graph paper provided or by drawing a neat sketch graph. Indicate the feasible region clearly on your graph. [6]

(c) By plotting a suitable objective line, or otherwise, determine the optimal solution. [4]

(d) Interpret the optimal solution to this problem as a weekly production schedule, giving the maximum profit the firm could expect to make with this schedule. Say why it is not possible for them to make a higher profit with the given resources. [4]

(e) Comment briefly on the validity of of the assumptions of proportionality, additivity and divisibility with regard to this particular problem. [4]

Question 3

The vertices $v_1, v_2, \ldots, v_n$ in a network N represent n sites. Each pair of sites can be joined by a direct link at a positive cost proportional to a weight $w(v_i, v_j)$ associated with the edge v_iv_j in N. It is required to construct a subnetwork T of N such that every pair of vertices of N are connected by a path in T and the total weight on the edges of T is as small as possible.

(a) Why is the required subnetwork T a *spanning tree* of N? [3]

(b) Describe the steps in *Prim's Algorithm* for finding a minimum weight spanning tree in a network. [4]

(c) The table below gives weights on the edges of a network N with vertices $v_1, v_2, \ldots, v_8$.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_1	-	3	5	4	2	4	1	6
v_2	3	-	3	6	5	1	4	4
v_3	5	3	-	3	4	5	2	1
v_4	4	6	3	-	6	7	2	5
v_5	2	5	4	6	-	5	8	3
v_6	4	1	5	7	5	-	6	7
v_7	1	4	2	2	8	6	-	5
v_8	6	4	1	5	3	7	5	-

(i) Describe how you would implement Prim's algorithm on this table of weights to find a minimum weight spanning tree in N, starting from the vertex v_1. Explain how you would select the next *two* vertices and edges to add to the tree and how you would modify the table of weights at each step. Complete the solution by Prim's algorithm (without further explanation) and draw the tree you obtain. State its total weight. [12]

(ii) Explain why there is just one alternative optimal solution, and draw the corresponding minimal spanning tree. [6]

Question 4

An oil company produces three grades of petrol, called super, regular and economy, by mixing two crude oils, C_1 and C_2. The amount of each crude oil required to make one barrel of petrol of each grade is given in the following table.

	Super	*Regular*	*Economy*
Crude oil C_1	0.7	0.5	0.4
Crude oil C_2	0.3	0.5	0.6

The company is opening a new refinery with a maximum production capacity of 10,000 barrels per week. There are at most 6000 barrels of C_1 and at most 5000 barrels of C_2 available to the refinery each week. The company estimates it can make a profit per barrel of £35 on the sale of super grade, £25 on regular grade and £20 on economy grade petrol. It wants to know the number of barrels of each type of petrol it should produce per week at this refinery to maximise its profit.

(a) Explain briefly how this problem can be modelled by the following linear programming problem. You should say what the decision variables represent and how the objective function and constraints are derived. [5]

Find $x_1, x_2, x_3 \in \mathbf{R}$ to maximize $z = 35x_1 + 25x_2 + 20x_3$ subject to:

$$0.7x_1 + 0.5x_2 + 0.4x_3 \leq 6000 \quad (0.1)$$

$$0.3x_1 + 0.5x_2 + 0.6x_3 \leq 5000 \quad (0.2)$$

$$x_1 + x_2 + x_3 \leq 10000 \quad (0.3)$$

$$x_1, x_2, x_3 \geq 0. \quad (0.4)$$

(b) Describe briefly the main assumptions that must be made for this model to be valid. [5]

(c) Convert the constraints to a suitable form for the solution of this linear programming problem by the simplex algorithm. Give the initial basic feasible solution. (*Note: you are NOT required to perform any iterations of the simplex algorithm in this question.*) [3]

(d) After several iterations of the simplex algorithm, the following tableau is obtained, where x_5, x_6, x_7 are the slack variables added to the constraints (0.1), (0.2) and (0.3) respectively.

Eq #	z	x_1	x_2	x_3	x_4	x_5	x_6	RS
0	1	0	0	2.5	25	0	12.5	280000
1	0	1	0	−0.5	5	0	−2.5	5000
2	0	0	0	0	1	1	−1	1000
3	0	0	1	1.5	−5	0	3.5	5000

(i) Explain why the solution represented by this tableau is optimal. [2]

(ii) Give the values of the decision variables that give an optimal solution for this problem and give the maximum value of z. Interpret this as a weekly production schedule for this refinery and give the maximum weekly profit they can expect to make on the sale of petrol. [4]

(iii) Give the values of the slack variables at the optimal solution. What extra information about the optimal solution do these values give? [4]

(iv) Suppose the refinery has an obligation to satisfy a weekly order for 1000 barrels of economy grade petrol and that it must meet this demand within its current weekly level of resources and production capacity. Would this cause their weekly profit to increase, decrease or stay the same? Give a reason for your answer. [2]

Question 5

Consider the following linear programming problem P.

Find $x_1, x_2 \in \mathbf{R}$ to minimize $z = x_1 + 2x_2$ subject to

$$\begin{aligned} x_1 - 2x_2 &\leq 2 \\ 2x_1 + x_2 &\geq 10 \\ x_1, x_2 &\geq 0. \end{aligned}$$

(a) Express P as a maximization problem and prepare the constraints for solution by the simplex algorithm, introducing slack, surplus and artificial variables, as appropriate. [3]

(b) Give the augmented initial basic feasible solution to the revised problem. [2]

(c) Modify the objective function for the solution of P by the Big M method. Express the revised objective function in terms of the variables that are non-basic in the solution you gave in part (b) above. [3]

(d) State a rule for determining (i) the entering variable (EV) and (ii) the corresponding leaving variable (LV) in the iterative step of the simplex algorithm. Show how your rules are used to determine the first EV and the corresponding LV in this problem. [5]

(e) Complete the solution of P by the Big M method. State the optimal solution to P and the maximum value of z. [12]

Question 6

(a) A local health authority must determine the allocation of its ambulance fleet for the following year. It costs £5000 per year to run an ambulance. The area covered by the authority is divided into two districts. Let x_1, x_2 be the number of ambulances assigned to district 1 and district 2 respectively (where fractional numbers represent an ambulance that is shared between the two districts and may operate in either). The average response time (in minutes) for an ambulance is $40 - 3x_1$ in district 1 and $50 - 4x_2$ in district 2. The local authority has three goals, which we list according to their priority:

- Goal 1: At most £100,000 per year should be spent on the ambulance service;
- Goal 2: Average response time in district 1 should be at most five minutes;
- Goal 3: Average response time in district 2 should be at most five minutes.

A solution to the problem is presented in the attached spreadsheet.

(i) Describe the model used. [7]

(ii) Describe the solution. [6]

(b) Your department needs to decide which of two secretarial candidates to hire. You have identified that three objectives are important to the decision: personality, typing ability and intelligence. You have also assessed the pairwise comparison matrix for these three objectives and the score of each candidate on each objective. Suppose you followed the Analytic Hierarchy Process and modelled the problem as shown in the spreadsheet attached.

(i) Describe the approach which was taken and its basic assumptions. [7]

(ii) Describe the results and their implication. [3]

(iii) What is the main limitation of this analysis? [2]

Question 7

A company is considering marketing a new product. There are initially three possibilities: (A) to market it immediately, (B) to abandon the project, (C) to employ a market research organisation to carry out a survey on the likely demand for this product.

The company is undecided about which choice to make because it does not know whether demand will be heavy (H), moderate (M) or light (L). If demand is heavy the return will be 100, if moderate 50 and if light 10. (All units are in thousands of pounds). Marketing costs 30. The chances of H and M are assessed as 0.2 and 0.4 respectively.

The market research organisation can produce a favourable (f), neutral (n), or unfavourable (u), report. The company makes the following assessments:

$$\begin{array}{lll} p(f|H) = 0.9 & p(f|M) = 0.2 & p(f|L) = 0.1 \\ p(u|H) = 0 & p(u|M) = 0.1 & p(u|L) = 0.6 \end{array}$$

After the report of the survey has been received the company can choose between (A) and (B) above.

The cost of the survey is S.

(a) Draw an appropriate decision tree for this problem. [19]

(b) What is the maximum value of S that the company would be prepared to pay for the survey? [3]

(c) If $S = 3$ and the company employs the market research organisation what is the expected return overall if the report is favourable? [3]