UNIVERSITY OF LONDON

GOLDSMITHS COLLEGE

B. Sc. Examination 2002

MATHEMATICS

MT51008A (M151) Calculus and Mathematical Methods

Duration: 3 hours

Date and time:

Do not answer more than THREE questions in section A.

Do not answer more than THREE questions in section B.

Full marks will be awarded for complete answers to SIX questions; three from section A and three from section B.

Tables of integrals to be provided.

Electronic calculators may be used. The make and model should be specified on the script and the calculator must not be programmed prior to the examination.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

SECTION A

Question 1 (a) Using the Sandwich theorem, or otherwise, prove that:

$$\lim_{x\to 0} \frac{x \sin 2x}{x^2 - 1} = 0.$$

[5]

(b) Find each of the following limits.

(i) $\lim_{x\to 0} \dfrac{e^x - 1}{x}$ [3]

(ii) $\lim_{x\to 0} \dfrac{\sec x - 1}{1 - \cos x}$. [4]

(c) (i) Sketch the graphs of $y = 3|x+2|$ and $y = 2 - x$ on the same axes. [2]

(ii) Find, algebraically, the solution set for the following inequality,

$$3|x+2| < 2 - x.$$

Express your answer as an interval and give the supremum of that interval. [6]

Question 2 (a) By using the method of Implicit Differentiation, or otherwise, find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y at all points at which this derivative exists on the curve

$$2x^2 - 2xy + 3y^2 = 15.$$

[4]

Determine all points (x, y) on this curve where $\frac{\mathrm{d}y}{\mathrm{d}x}$ does not exist. [3]

(b) Define

$$f(x) = \begin{cases} \dfrac{\log_e(1+x)}{x} & \text{if } x \neq 0; \\ 1 & \text{if } x = 0. \end{cases}$$

(i) Show that $f'(x) = \dfrac{x - (1+x)\log_e(1+x)}{x^2(1+x)}$ for $x \neq 0$; [3]

(ii) Using l'Hôpital's rule, or otherwise, show that

$$\lim_{h\to 0} \frac{\log_e(1+h) - h}{h^2} = -\frac{1}{2}.$$

Hence deduce that $f'(0) = -\frac{1}{2}$. [5]

(iii) Show that $f'(x)$ is continuous at $x = 0$. [5]

Question 3 (a) State the Mean Value Theorem for a function $f\colon [a,b] \to \mathbb{R}$. [4]

(b) Apply the Mean Value Theorem to the function $f(x) = \sinh x$ on the interval $[a,b]$, where $0 \le a < b$, and deduce that

$$\sinh b > b$$

for all $b > 0$. [5]

You may assume that $\cosh c > 1$ *for all* $c > 0$.

(c) The function $f\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = (x+1)(x-5)^2.$$

(i) Find any local maxima and any local minima. [5]

(ii) Find any points of inflection. [2]

(iii) Sketch the graph of $y = f(x)$ showing the above information. [4]

Question 4 (a) State Taylor's Theorem, with an error term, for $f\colon \mathbb{R} \to \mathbb{R}$ which is $(n+1)$-times differentiable on $\mathbb{R}$. [5]

(b) Determine up to the coefficient of x^3 the Taylor expansion of the function $f(x) = \sin x$ about $x = 0$, and find an expression for the error term. [7]

By using the fact that for any $t \in [0, 0.1]$ we have

$$0 \le \sin t \le t,$$

or otherwise, show that the worst possible error that might occur for $x \in [0, 0.1]$ is not more than 4.2×10^{-6}. [3]

(c) Show that

$$\lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n = \frac{1}{e}.$$

You may use the identity

$$\lim_{n\to\infty} \left(1 + \frac{\alpha}{n}\right)^n = e^\alpha.$$

Hence deduce that

$$\sum_{n=0}^{\infty} \left(\frac{x}{n+1}\right)^n$$

converges for all $x \in \mathbb{R}$. [5]

SECTION B

Question 5 (a) Determine the following indefinite integral

$$\int e^{-2x} \cos 3x \, \mathrm{d}x.$$

[10]

(b) Sketch the area over which the following integral is evaluated

$$\int_{x=0}^{x=1} \int_{y=x^2}^{y=\sqrt{x}} \left(x^2 + 2y\right) \, \mathrm{d}y \mathrm{d}x.$$

Evaluate this double integral. [10]

Question 6 (a) Evaluate the following in the form $x + iy$

(i)

$$(5 - 3i)\,(-2 - 5i)\,;$$

(ii)

$$\frac{-2 + 5i}{4 - 3i}.$$

[4]

(b) Find in the form $re^{i\theta}$, where $-\pi < \theta \leq \pi$, the complex number

$$\left(\sqrt{3} + i\right)^8.$$

[4]

(c) Use De Moivre's Theorem to show that

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

[4]

(d) Solve the equation

$$z^4 = \left(1 + \sqrt{3}i\right)^3,$$

where $z = x + iy$, obtaining four distinct roots. [8]

Question 7 (a) Sketch the region bounded by the parabola $y = 5 - x^2$ and the line $y = 3 - x$, and find the volume of the solid generated when this region is rotated about the x-axis. [10]

(b) Let $I_n = \int x^n \cos x \, dx$.

(i) Show that

$$I_n = x^n \sin x + nx^{n-1} \cos x - n(n-1) I_{n-2}, \quad \text{for} \quad n \geq 2.$$

(ii) Determine the indefinite integral I_n for $n = 4$. [10]

Question 8 (a) Solve the following first order differential equation

$$\left(x^2 + 1\right) \frac{dy}{dx} + 2xy = 4x^2,$$

given that $y = 0$ when $x = 1$. [10]

(b) Find the general solution of the following second order differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 3x^2 + 2.$$

[10]

Question 9 (a) Given $f(x, y) = x^2y + xy^2$, where $x = u + v$ and $y = -u + v$, sketch appropriate tree diagrams and determine

$$\frac{\partial f}{\partial u} \text{ and } \frac{\partial f}{\partial v} \text{ in terms of } u \text{ and } v.$$

[10]

(b) The Luminosity L, radius R, and temperature T of a star are related by the expression

$$L = CR^2T^4,$$

where C is a constant. Suppose that there are percentage errors of 10% and 5% in the measurements of R and T respectively.

Approximate the maximum percentage error to first order in the calculated value of L.

[10]