UNIVERSITY OF LONDON

GOLDSMITHS COLLEGE

B. Sc. Examination 2002

MATHEMATICS

MT51007A (M150) Discrete Mathematics, Vectors and Matrices

Duration: 3 hours

Date and time:

Do not attempt more than SIX questions from Section A and THREE questions from Section B. Full marks will be awarded for complete answers to SIX questions from Section A and THREE questions from Section B.

Each question in Section A carries 10 marks and each question in Section B carries 20 marks. There are 120 marks available on this paper.

Begin each question on a new page.

USE DIFFERENT ANSWER BOOKS FOR SECTION A AND SECTION B.

Electronic calculators may be used. The make and model should be specified on the script. The calculator must not be programmed prior to the examination. Calculators which display graphics, text or algebraic equations are not allowed.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

SECTION A

Question 1 Let A, B, C be subsets of a universal set $\mathcal{U}$.

(a) Verify the associative law $A \oplus (B \oplus C) = (A \oplus B) \oplus C$ by constructing an appropriate membership table. [4]

(b) Draw a Venn diagram to illustrate the set $A \oplus (B \oplus C)$. [2]

(c) Let $S = \{1, -1\}$ and $T = \{x \in \mathbb{R} \, : \, (x^2 - 1)(x^2 + 1) = 0\}$.

(i) Prove that $S \subseteq T$.

(ii) Is $S = T$? Justify your answer.

[4]

Question 2 (a) Let X and Y be sets, and $f : X \to Y$ be a function. Explain what it means to say that f is *one-to-one* and that f is *onto*. [2]

(b) State whether or not each of the following functions is (i) one-to-one, and (ii) onto. Justify your answers by giving a counterexample when the function does not satisfy one of these properties.

$$f : \mathbb{R} \to \mathbb{Z} \text{ defined by } f(x) = \lfloor x \rfloor.$$

$$g : \mathbb{R} \to \mathbb{R} \text{ defined by } g(x) = e^x.$$

[6]

(c) Give the inverse function of the following function h.

$$h : \mathbb{R} \to \mathbb{R} \text{ defined by } h(x) = x^3 + 1.$$

[2]

Question 3 (a) Consider the sequence: 2, 5, 8, 11, 14, ...

(i) State a recurrence formula which expresses the nth term u_n in terms of the $(n-1)$th term u_{n-1}.

(ii) State, without proving, a closed formula which determines u_n for all $n \geq 1$.

[3]

(b) The terms of a sequence are given by putting $u_1 = 1$, and, for $n \geq 2$,

$$u_n = u_{n-1} + 2n - 1.$$

(i) Determine u_2.

(ii) Prove by induction that the closed formula $u_n = n^2$ holds for all $n \geq 1$.

[7]

Question 4 (a) Express the sum $1+2+4+\ldots+1024$ using sigma notation. (You are not required to calculate this sum.) [2]

(b) Evaluate $\sum_{r=6}^{30} 2r+1$ using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} 1$. [5]

(c) Express the binary number 1011.01 as a decimal number. [3]

Question 5 Let b, c and d be integers.

(a) (i) Explain what it means to say that d is a *divisor* of b.

(ii) Prove that if d is a divisor of b then d is a divisor of $bc + d$.

[4]

(b) Suppose $d \geq 1$.

(i) Explain what it means to say that b *is congruent to* c *modulo* d.

(ii) Determine the set S of all integers x such that $x \equiv 13 \pmod 7$. [3]

(c) Find integers x, y, z such that $0 \leq x, y, z \leq 6$ and

(i) $[13]_7 = [x]_7$

(ii) $[3]_7 \oplus [5]_7 = [y]_7$

(iii) $[4]_7 \otimes [z]_7 = [1]_7$

in $\mathbb{Z}_7$. [3]

Question 6 (a) An ordered sequence of four digits is formed by choosing digits without repetition from the set $S = \{1, 2, 3, 4, 5, 6, 7\}$. Determine:

(i) the total number of such sequences

(ii) the number of sequences which begin with an odd number

(iii) the number of sequences which end with an odd number

(iv) the number of sequences which begin and end with an odd number

(v) the number of sequences which begin with an odd number or end with an odd number or both

(vi) the number of sequences which begin with an odd number or end with an odd number but not both

(vii) the number of *subsets* of S with exactly four elements.

[7]

(b) (i) Expand $(x + y)^n$ using the binomial theorem.

(ii) Determine the coefficient of y^5 in $(1 - y)^{13}$.

[3]

Question 7 (a) Let $S = \{2, 3, 4, 5, 6, 7, 8\}$. Let p, q be the following propositions on $n \in S$.

p: n is a divisor of 12;

q: n is prime.

Describe, using the listing method, the truth set of each of the following propositions:

(i) p;

(ii) q;

(iii) $\neg p$;

(iv) $p \vee q$;

(v) $p \rightarrow q$;

(vi) $p \leftrightarrow q$.

[6]

(b) Let $T = \{1, 2, 3, 4\}$. We define a relation on T by, for any two elements x and y of T,

x is related to y if and only if $x \leq y$.

Draw the relationship digraph corresponding to this relation. [4]

Question 8 (a) Let H be the graph with the following adjacency matrix.

$$A(H) = \begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 1 & 1 & 1 & 1 \\ v_2 & 1 & 0 & 1 & 0 & 0 \\ v_3 & 1 & 1 & 0 & 1 & 0 \\ v_4 & 1 & 0 & 1 & 0 & 1 \\ v_5 & 1 & 0 & 0 & 1 & 0 \end{array}$$

(i) Draw H. [2]

(ii) Draw a spanning tree of H with degree sequence 3,2,1,1,1. [2]

(Take care to label the vertices as v_1, v_2, v_3, v_4, v_5 in your drawings.)

(b) Let G be a graph. Prove that

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|.$$

[3]

(c) Let T be a binary tree of height h.

(i) State, without proof, the number of vertices on level i of T for each i, $0 \leq i \leq h - 1$.

(ii) What is the total number of vertices on levels 0,1,2,...,$h - 1$?

[3]

SECTION B

Question 9 Let a and b be real numbers and let

$$A = \begin{pmatrix} 0 & 2 & -4 & -1 & a \\ -3 & -1 & 2 & -1 & b \\ 1 & -1 & 2 & 1 & 1 \end{pmatrix}.$$

(a) Find elementary matrices $E_1, E_2, \ldots, E_r$ such that $E_r E_{r-1} \ldots, E_2 E_1 A$ is in row echelon form. [12]

(b) Given the following system of equations

$$\begin{array}{rcrcrcrcl} & & 2x_2 & - & 4x_3 & - & x_4 & = & a \\ -3x_1 & - & x_2 & + & 2x_3 & - & x_4 & = & b \\ x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 1 \end{array}$$

with augmented matrix A, show that the system has a solution if, and only if, $2a + b + 3 = 0$. Further, find all solutions of the system for $a = b = -1$. [8]

Question 10 (a) Find the determinant $\det(A)$ of the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & -2 & 2 \\ 2 & -1 & -3 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

[5]

(b) Define an *invertible* matrix. [2]

Find the inverse of the following matrix:

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -2 \\ 2 & -1 & -3 \end{pmatrix}.$$

[7]

(c) Let A and B be invertible matrices and let A^T be the transpose of A. Show that both AB and A^T are invertible. [6]

Question 11 (a) Let V be a subspace of $\mathbb{R}^n$. Explain what it means for a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ in V to be:

(i) *linearly independent*,

(ii) a *spanning set* for V,

(iii) a *basis* for V. [5]

Let V be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{v}_1 = (1, 1, -1, 0)$, $\mathbf{v}_2 = (1, 3, 1, 2)$, $\mathbf{v}_3 = (3, 4, -2, 1)$ and $\mathbf{v}_4 = (1, 3, 1, 2)$. Find a basis for V and determine the dimension of V. [6]

(b) Let $\mathbf{v}_1 = (1, 2, 1), \mathbf{v}_2 = (-1, 1, 1)$ and $\mathbf{v}_3 = (1, 3, -2)$ be vectors in $\mathbb{R}^3$.

(i) Compute the triple product $\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)$. [3]

(ii) Determine the equation of the plane which is perpendicular to $\mathbf{v}_1$ and passes through the point $\mathbf{P}(1, -2, 2)$. [3]

(c) A particle moves in $\mathbb{R}^3$ along the path

$$\mathbf{r}(t) = 2t^2\,\mathbf{i} + \sqrt{20}\,t\,\mathbf{j} + t^2\,\mathbf{k}$$

where the parameter t denotes the time. Find the angle between the velocity $\mathbf{v}(t)$ and the acceleration $\mathbf{a}(t)$ of the particle at $t = 1$.

[3]

Question 12 Let A be an $n \times n$ matrix and let λ be a real number.

(a) Explain what is meant by saying that λ is an *eigenvalue* of A. [2]

(b) Find the eigenvalues and corresponding eigenspaces of the matrix

$$A = \begin{pmatrix} 1 & -2 \\ -4 & 3 \end{pmatrix}.$$

Find a basis for each eigenspace of A. [15]

Is A diagonalizable? Give a reason. [3]