UNIVERSITY OF LONDON

FOR EXTERNAL STUDENTS

B. Sc. Examination 2002

MATHEMATICS

M204 Linear Algebra

Duration: 2 hours 15 minutes

Date and time:

Do not attempt more than FOUR questions on this paper.

Full marks will be awarded for complete answers to FOUR questions.

Electronic calculators may be used. The make and model should be specified on the script. The calculator must not be programmed prior to the examination. Calculators which display graphics, text or algebraic equations are not allowed.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Question 1 (a) Let X be a vector space over a field $\mathbb{F}$ and let $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ and $\{\mathbf{y}_1, \ldots, \mathbf{y}_m\}$ be finite subsets of X.

(i) Define the set span $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$. [1]

(ii) Show that span $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a subspace of X. [5]

(iii) Explain what it means to say that $\{\mathbf{y}_1, \ldots, \mathbf{y}_m\}$ is a *linearly independent* set and that $\{\mathbf{y}_1, \ldots, \mathbf{y}_m\}$ is a *basis* of X. [2]

(iv) Suppose that $\{\mathbf{y}_1, \ldots, \mathbf{y}_m\} \subseteq \operatorname{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$. Describe how a matrix A may be formed such that if $\mathbf{z} = \mu_1\mathbf{y}_1 + \cdots + \mu_m\mathbf{y}_m$ then $\mathbf{z} = \lambda_1\mathbf{x}_1 + \cdots + \lambda_m\mathbf{x}_n$ where

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = A \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix}.$$

Deduce that if $\{\mathbf{y}_1, \ldots, \mathbf{y}_m\}$ is linearly independent then $m \leq n$. [9]

(v) Conclude that if $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ and $\{\mathbf{y}_1, \ldots, \mathbf{y}_m\}$ are bases of X then $n = m$. [2]

(b) Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and $\mathbf{y}$ be vectors in the space $\mathbb{R}^3$ given by

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} 4 \\ 6 \\ 5 \end{pmatrix}.$$

(i) Is $\mathbf{z} \in \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$? [4]

(ii) Is $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ a basis of $\mathbb{R}^3$? Justify your answer. [2]

Question 2 (a) Let X be a vector space over a field $\mathbb{F}$.

(i) State necessary and sufficient conditions for a subset Y of X to be a *subspace* of X. [3]

(ii) Let U and V be subspaces of X. Define $U \cap V$ and $U + V$. [2]

(iii) Prove that $U + V$ is a subspace of X. [5]

(b) Let X be the vector space $\mathbb{R}^{\mathbb{R}}$ of all real valued functions on $\mathbb{R}$ and let Y be the subset of X of all continuous functions. Show that Y is a subspace of X. [6]

(c) Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ and $\mathbf{v}_1, \mathbf{v}_2$ be vectors in the space $\mathbb{R}^4$ given by

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} 2 \\ -3 \\ 5 \\ 1 \end{pmatrix}, \ \mathbf{u}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \ \mathbf{v}_1 = \begin{pmatrix} 2 \\ 2 \\ -1 \\ 5 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix}.$$

Set $U = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

(i) Find bases for U, V, $U + V$ and $U \cap V$. [8]

(ii) What are $\dim U$, $\dim V$, $\dim U + V$ and $\dim U \cap V$? [1]

Question 3 (a) Let X be a finite dimensional vector spaces over a field $\mathbb{F}$ and let $T: X \to X$ be a linear transform.

(i) Define what it means for T to be a linear transform. [2]

(ii) Define $\operatorname{Ker} T$, the kernel of T, and $\operatorname{Im} T$, the image of T. [2]

(iii) Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ be a basis of $\operatorname{Ker} T$ and let $\{\mathbf{u}_1, \ldots, \mathbf{u}_r, \mathbf{v}_1, \ldots, \mathbf{v}_s\}$ be an extension to a basis of X. Show that $\{T(\mathbf{v}_1), \ldots, \mathrm{T}(\mathbf{v}_s)\}$ is a basis of $\operatorname{Im} T$. [9]

(b) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - 3y \\ 2y + z \\ x + 3y + 3z \end{pmatrix},$$

and let S be the standard basis.

(i) Compute the matrix representing T with respect to S. [2]

(ii) Give a basis of $\operatorname{Ker} T$. [3]

(iii) Find a basis B of $\mathbb{R}^3$ such that the matrix $D = (T)_{B,B}$ representing the transform T is diagonal.
You do not need to find a matrix P such that $D = P^{-1}(T)_{S,S}P$. [7]

Question 4 (a) Let X be a vector space over a field $\mathbb{F}$ and let $T\colon X \to X$ be a linear transform.

Define the terms *eigenvalue*, *eigenvector* and *eigenspace* of T. [3]

(b) Let

$$M = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

(i) Find the eigenvalues and corresponding eigenspaces for M when M is considered as a linear transform acting on $\mathbb{R}^2$. [7]

(ii) Find an invertable matrix P, its inverse P^{-1} and a diagonal matrix D such that $D = P^{-1}MP$. [4]

(iii) Calculate the matrix M^{10}.

You may note that $3^{10} = 59049$. [2]

(c) Let P_2 be the real vector space of all polynomials of degree less than or equal to 2, with real coefficients, and basis $B = \{1, x, x^2\}$. Let $T\colon P_2 \to P_2$ be defined by

$$T(p) = \frac{\mathrm{d}^2 p}{\mathrm{d}x^2} - 3\frac{\mathrm{d}p}{\mathrm{d}x} + p.$$

(i) Show that T is a linear transform. [3]

(ii) Compute the matrix representing T with respect to B. [3]

(iii) Use the matrix representation of T to find a particular integral $p \in P_2$ of the differential equation

$$\frac{\mathrm{d}^2 p}{\mathrm{d}x^2} - 3\frac{\mathrm{d}p}{\mathrm{d}x} + p = x^2 - 5x.$$

[3]

Question 5 (a) Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ be an orthonormal set of vectors in $\mathbb{R}^n$ and $U = \operatorname{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$.

(i) Define what it means for $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ to be orthonormal. [2]

(ii) By using the fact that $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is orthonormal, show that the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is linearly independent.
Hence deduce that $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is a basis of U. [4]

(iii) Define the *orthogonal compliment* $U^\perp$ of U. [1]

(iv) Show that if $\mathbf{y}$ is orthogonal to each of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ then $\mathbf{y} \in U^\perp$. [4]

(b) Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{x}$ be vectors in $\mathbb{R}^4$ and A be the 4×2 matrix given by

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ -2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & -2 \end{pmatrix}$$

Set $U = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

(i) Find an orthonormal bases of U. [3]

(ii) Find the least squares solution to the system of equations $A\mathbf{v} = \mathbf{x}$, and hence find the closest point $\mathbf{u} \in U$ to $\mathbf{x}$. [5]

(iii) Find a basis of $U^\perp$, and hence extend your orthonormal basis of U to an orthonormal basis of $\mathbb{R}^4$. [6]