Analytical properties of discrete planes

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Some classes of medical images can be represented by meshes of discrete polygons that are portions of discrete planes, while the polygon edges are segments of discrete lines. Classically, the discrete planes and lines are defined algorithmically. While being quite satisfactory regarding various practical purposes, these definitions are not always easy to use for obtaining deep structural results. Moreover, storing big volumetric images might be problematic. This may happen, for instance, if a 3D object is represented "slicewisely" (similar to the "Visible Human" representation). Furthermore, sometimes it may be non-trivial to perform certain elementary image processing operations, such as verifying if a voxel (or a set of voxels) belong to a discrete plane, or to the intersection or the union of several discrete planes, etc.

A promising approach which may help overcome some of the above mentioned difficulties is the one based on the analytical description of an object. In 1991 Reveilles proposed an analytical definition of a discrete straight line [1], which extends to a discrete plane. According to it, a discrete line L(a, b, μ, ω) is the set of integer points satisfying a double linear Diophantine inequality of the form $0 <$ ax + by + $\mu < \omega$. Here $\mu \in \mathbb{Z}$ is an internal translation constant measuring the shift of the line with respect to the origin, while $\omega \in \mathbb{N}$ is the arithmetic thickness of the line. Respectively, a discrete plane P(a, b, c, μ, ω) is a set of integer points with $0 <$ ax+by+cz+$\mu < \omega$, where the parameters $\mu \in \mathbb{Z}$ and $\omega \in \mathbb{N}$ have similar meaning. L(a, b, μ, ω) and P(a, b, c, μ, ω) can be regarded as a discretization of a line (resp. plane) with coefficients a, b (resp. a, b, c). It can be shown that if $\omega = \max(|a|, |b|)$ (resp. $\omega = \max(|a|, |b|, |c|)$, the above definitions are equivalent to other well-known classical definitions of lines and planes (see [2] for getting acquainted with different approaches to defining digital straightness, and [3] for a study on digital flatness).

The main advantage of an analytical definition seems to lie in the fact that one can study an object in terms of a few parameters that define it. Along with ensuring a very compact object encoding, this may significantly facilitate the geometric and analytic reasoning and help describe theoretical results in a more rigorous and elegant form. For example, one can easily show that a discrete plane P(a, b, c, μ, ω) has tunnels if and only if $\omega < \max(|a|, |b|, |c|)$. Analytic definitions may also help raise new theoretical questions, whose rigorous formulation would be difficult by other means. For instance, given three integers (plane coefficients) a, b, and c, one can look for the maximal value ω for which the discrete plane P(a, b, c, μ, ω) is disconnected, *i.e.*, one can define plane connectivity number as Ω(a, b, c) = $\max\{\omega$: the discrete plane P(a, b, c, μ, ω) is disconnected$\}$. Thus the notion of discrete plane connectivity can be properly formalized and studied. This problem is important since, on the one hand, discrete plane is a very fundamental primitive in volume modeling (in particular, in medical imaging) and properties of digital flatness are of a wide interest from theoretical perspective. On the other hand, connectivity is a principal topological characteristic, crucial for the deeper understanding the properties of a given class of objects and, possibly, for designing new more powerful visualization techniques.

Discrete plane connectivity, however, cannot be characterized by a condition as simple as the one above characterizing tunnel-freedom. To our knowledge of the available literature and according to our personal communications, this problem is still open, although within the last ten years or more, several researchers (including Reveilles, among others) have attempted to resolve it. So it becomes a challenge to achieve certain progress towards its solution.

With this talk we will present a solution to the discrete plane connectivity problem in terms of the analytical discrete plane definition. In some cases the problem admits an explicit answer. Thus, for example, we show that if the plane coefficient satisfy the inequality $c > a + 2b$, then $\Omega(a, b, c) = c-a-b+gcd(a, b)-1$. One can also determine other classes of instances admitting equally simple answer. In other cases the solution may be more complicated.

Our approach is based on study of various properties of discrete planes, some of which may be of interest in their own. We believe that some of these properties may not only serve as a theoretical background for constructing new discretization algorithms, but might also have direct application to designing sophisticated methods that will allow one to simultaneously visualize and study the surface of a given human organ and its interior.

References:

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3. Brimkov, V.E., Notes on digital flatness. TR 2002-01, Laboratory on Signal, Image and Communications, UFR SFA, University of Poitiers, France, 2002, 52 pages.