INTERIOR METRIC SHORTEST PATHS AND LOOPS IN RIEMANNIAN MANIFOLDS WITH NOT NECESSARILY SMOOTH BOUNDARY

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PREFACE and INTRODUCTION

This paper wants to discuss the foundation of an approach to an intrinsic geometry in Riemannian manifolds with boundary. That means, we wish to give a base for studies on shortest paths and intrinsical distance (which is defined for any two points as length infimum of connecting paths); we aim at finding out relations to global topological properties of the manifold. This paper contains proofs of some of the results announced by us in [31] and [32] . Except from the recent independent work of R. and S. Alexander, R.L. Bishop and I. D. Berg and several other geometers in contact with this group, where we learned about by private communications and by [29] there does not seem to exist any systematic treatment of our topic. This omission is astonishing since manifolds with boundary are so important for topologists as well in global as in local considerations. The absence ist even more surprising if one remembersthat on one hand intrinsic Riemannian geometry has been useful for problems in science, technics and economy and that one the other hand just in technics and economy various extremal problems are posed together with inequalities and boundary constraints so one might think that Riemannian manifolds with boundary could serve as unseful models for a description of many of those problems. For instance the main subject of this paper - a shortest path in a homtopy class - may be visualised as a rope tightly stretched over obstacles perhaps partly wrapped around extrusions.

Of course, there also should exist application possibilities for problems in differential geometry itself. First methods and results with similar intentions however more developed than in this paper might be useful to study global topological properties of certain open non complete manifolds which have been resistant against global investigations until now. We have in mind Riemannian manifolds which are weak deformation retract of their Cauchy completion. One may perhaps study global questions easier using the existence of shortest paths, loops and eventually a generalized concept of cut loci only available in the completion which may be a bordered manifold treatable by our tools.

Of course, it is desirable that an eventually new concept will have relations and will contribute to understand still existing classical problems. As far as we know there are at least two places where intrinsical ad hoc considerations for manifolds with boundary have been used in the proofs of major theorems; those are the theorem of Efimov [19] and the theorem of Cohn-Vossen i.e., the generalisation of the Gauss-Bonnet theorem to complete surfaces [30] . (In [19] p. 535-541 J. Milnor proves a certain two dimensional version of our result on p. 47 the proof given there cannot be generalized to higher dimensions.

We believe that those examples should not be the only places for the application of intrinsic geometrical considerations on manifolds with boundary. To underline this we will finally speculate - and what we say now has to be relativated in this sense - about eventual application possibilities on a much stdied class of examples being of interest for a large group of geometers. It can be shown that if a simply connected metrically complete surface with C^2 -smooth boundary curve is subsurface of some simply connected complete two dimensional Riemannian manifold without conjugate points, then the subsurface has a C^1-smooth intrinsic distance function with locally Lipschitz continuous gradient in the interior of the subsurface. (More subtle considerations are possible.) This result may eventually be the base to distinguish certain bordered minimal surfaces by means of their intrinsic geometric properties for instance by studying relations between boundary curve and shape of distance circles i.e. curvature number of connected components, gradient lines of the intrinsic distance function e.t.c. The condition of being a minimal surface may eventually force certain intrinsic properties. Since we hope that it is possible to develope a generalized concept of cut-loci for manifolds with boundary there may then eventually come out a helpful tool to distinguish between topological types of certain minimal surfaces.

We describe now roughly content and construction of our paper. If one defines as usually on a Riemannian manifold the distance between any two points by the infimum of arclengths of piece wise C^1 -smooth paths connecting the points-(where the arclength is defined as the integral of the tangent vectors norm)- then this distance makes the manifold into a metric space whose topology aggrees with the manifolds topology. The same holds as we will show in II for Riemannian manifolds with boundary, if the boundary fullfils certain conditions. Those metric spaces are examples for metric spaces with an interior metric in the sense of W. Rinow and various results for Riemannian manifolds can be proved without the presence of a smooth structure simply within the frame of such metric spaces. What is a space with an interior metric? It is a metric space (M, d) which satisfies certain conditions. We need some explanations to describe those conditions. In a metric space (M,d) we can define the length of a path $c: I \to (M,d)$ as the supremum of sums of distances between partition points taken over all finite partitions of the parameter intervall I. If this supremum is finite the path is called rectifiable. Now if any two points x, y in (M,d) can be joined by a rectifiable path then one may define a new metric $\tilde{d}(,)$ by taking for the new distance between x and y $\tilde{d}(x,y)$ as infimum of lengths over all rectifiable paths connecting x and y. If the metrics d and $\tilde{d}$ agree we say that (M,d) is a metric space with an interior metric. Note in general even the topologies induced by the metrics d and $\tilde{d}$ on M need not agree. Take for instance as space (M,d) the metric subspace of the eucledean plane got from the union of segments connecting the origin with the point $(1,0)$ and with all points of the sequence $(1,\frac{1}{n})$ $n \in \mathbb{N}$; here the points are given in eucledean coordinates. Now the space (M,d) is compact but not in the metric $\tilde{d}$ since in $(M,\tilde{d})$ the sequence $(1,\frac{1}{n})$ has no clusterpoint. Further also if the topologies induced by d and $\tilde{d}$ on M coincide the metrics need not agree take for (M,d) the real numbers with the metric

$d(x,y) := \frac{|x-y|}{1+|x-y|}$, $|x|$ denotes the abolute value of the real number x.

In this example(*) (M,d) and the corresponding $(M,\tilde{d})$ are both complete metric spaces which are homeomorphic but not isometric.
Our use of the word metric in this paper may be ambiguous since we use it for the distance funtion $d(,)$ if it defines a metric on some set as well as for the such defined structure. If we speak of a Riemannian metric or Riemannian structure we of course always think as it is common of a form inducing scalarproducts on the tangent spaces.

(* In this space (M,d) shortest paths are <u>not</u> distance realizing.)

The paper is divided in three chapters and an appendix. In the first chapter we only discuss results which hold for metric spaces with an interior metric independent of the existence of a smooth structure. The results we get in the first chapter are in the third chapter applied to get the existence part of several results given in III_3. One of the main intentions in the second chapter is to show that this application is possible and to clarify the relations between the metrical space structure and the manifold structure on the Riemannian manifolds with boundary which we describe in II_2 and II_5. Those clarifcations provide also several facts to be used in the proofs in III_1 and III_2.
In the first chapter much weight is put on the question: When is a metric space Heine-Borelsch i.e. when does the space have the property that his bounded and closed subsets are compact?. We prove in I_1 that a metric space with an interior metric is Heine Borelsch iff it is locally compact and complete. The importance of the Heine-Borel property lies the fact that it assures the existence of distance realising paths in metric spaces with an interior metric. Even if we would only discuss compact spaces, if we are interested in shortest loops in homotopy classes the Heine-Borel property is yet important. Since it is comfortable to show the existence of shortest loops in a homotopy class by proving the existence of corresponding shortest paths in the universal convering space and because the covering space needs not to be compact but will still be Heine-Borelsch if the covered space is Heine-Borelsch with an interior metric.

All our results in III those are mainly existence theorems for shortest paths and loops with certain smoothness qualities hold for n-dimensional bordered manifolds which are locally diffeomorphic to a convex set in E^n. However convexity is no invariant property under diffeomorphisms. Since we wish to investigate intrinsic geometric questions we have to present a concept using intrinsic definitions for those types of manifolds we discuss. That means we must describe intrinsically what properties a set in E^n corresponding to a coordinate neighbourhood in the manifold must have and those properties must be invariant under a coordinate exchange. We think that the definition we give in II_1 by manifolds satisfying a cone condition is appropriate for this purpose, since it allows to define all later used objects and their properties without any extendebility assumption beyond the manifold.

After the preparations in the second chapter we prove in III certain smoothness properties of shortest rectifiable paths in Riemannian manifolds with boundary, which are locally diffeomorphic to a convex set in R^n. One of the results given in III, p. 47 can be formulated as follows: If M is a C^1-smooth differentiable manifold with boundary being locally C^1-diffeomorphic to a convex set in E^n and if M carries a Riemannian metric g which has locally Lipschitz continuous local representations (g_{ik}) then every shortest rectifiable path c given in arclength parametrisation $c: [o,a] \longrightarrow M$, is C^1-smooth on the open intervall (O,a). (With an additional argument not included in this paper we have that c is C^1 on all $[o,a]$).
There exist independently proved results with similar and different intentions which were given by R. and S. Alexander in [23] and by R. L. Bishop private in communications. The methods used by them are different from ours. We show in III_2 the smoothness

properties of the shortest paths by applying the calculus of variations to rectifiable curves. We justify this approach by proving in III_1 that for an absolutely continuous path $c: J \rightarrow (M,d)$, — (with (M,d) the metric space related to (M,g)·)—, the arclength definition via the integral of the tangent vectors norm, which we will call shortly 'differentiable length' and denote by $L_D(c)$ agrees with the so called 'rectifiable length' denoted by $L_R(c)$ which is defined as the supremum of sums of distances between partition points over all finite partitions of the parameter intervall I. This probably per se interesting result, which even holds, if the Riemannian metric g is only continuous, allows us to say, that a rectifiable shortest curve in arclength parametrisation has less or equal energy than the test-curves used in the variation in III_2. The proof in III_2 is locally performed in a subset of E^n corresponding to a coordinate neigbourhood, the proof consists in showing that all difference quotients of a shortest path's derivative have a common bound in the L^2-norm. During this proof we essentially exploit the assumption that the manifold M is locally diffeomorphic to a convex set in R^n. Namely, since the variation test-curves are convex combinations of points of the given shortest path, they are admissable since they stay in the due to our assumption convex chosen set in R^n related to the coordinate neighbourhood. We explain the background analysis used in the variational calculus in the appendix.

It may be notable that the theorem in III_3 assuring the existence of C^1 smooth distance realising paths in a metrically complete manifold described above-(which may be viewed as a generalisation of a part of the theorem of Hopf and Rinow)- as well as all other results in this paper are proved without the differential geometric standard tools like Gauss-lemma, exponential map e.t.c..

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I METRICAL SPACES WITH INTERIOR METRIC

I₁ When is a metrical space Heine-Borelsch?

Definition: In a general metric space (M,d) we can define the length $L(c)$ of a continous curve $c:I \to M$, $I = [a,b]$, by $L(c) := \sup\{c(Z) \mid Z \in \mathcal{B}(I)\}$ with $Z = (a=t_0,\ldots,t_n=b)$ a finite partition of I, $c(Z) := \sum_i d(c(t_i),c(t_{i+1}))$ and $\mathcal{B}(I)$ denotes the set of all finite partitions of I. If $L(c)$ is finite the curve is called rectifiable. If for all points $p,q \in M$ we have

$d(p,q) = \inf\{L(c) \mid c$ rectifiable continous curve from p to q$\}$

the metric space is called a space with an interior metric.

The following immediate proposition contains an essential property of spaces with an interior metric. It will be used in the basic theorem later on.

Proposition : Let (M,d) be a space with an interior metric. Further let x_0 be an arbitrary point in M and $S(x_0,r) := \{x \mid d(x_0,x)=r\}$ a distance sphere around x_0 with arbitrary radius r. Now assume $y \in M$ is an arbitrary point with $d(x_0,y) > r$. Then for any given $\epsilon > 0$ there exists some $x' \in S(x_0,r)$ such that $d(x',y) \leq d(x_0,y) - r + \epsilon$.

Proof : Since (M,d) has an interior metric there exists a curve $c:[a,b] \to M$, $c(a)=x_0$, $c(b)=y$ with $L(c)<d(x_0,y)+\epsilon$. The function $d(x_0,c(.)):[a,b] \to R$ is continous, thus we have $t' \in [a,b]$ with $d(x_0,c(t'))=r$. Using again the definition of an interior metric we get from this

$$d(x_0,c(t'))+d(c(t'),y) \leq L(c_{/[a,t']}) + L(c_{/[t',b]}) = L(c) \leq d(x_0,y)+\epsilon .$$

This yields $d(c(t'),y) \leq d(x_0,y) - r + \epsilon$.

Chosing $x' = c(t')$ we get our claim.

We come now to the basic theorem, which we present here with a selfcontained proof, because the proof of Rinow ([27],p.174) is part of broader considerations and the one of Busemann ([9],p.4) seems to contain a gap (see remark below).

Theorem : ([9], [10], [27]) A locally compact metrical space with an interior metric is Heine-Borelsch if and only if its metric is complete.

(Heine-Borelsch means that bounded and closed sets are compact.)

Proof: A metrical space beeing Heine-Borelsch is obviously locally compact and complete. The difficulty consists in proving the other direction. For this purpose let z be an arbitrary point in M. Further be $s = \sup\{r \mid K(z,r) \text{ is compact}\}$, here as always in this proof $K(z,r):=\{x \mid d(x,z) \leq r, x \in M\}$ denotes a distance full-ball.

First, because of the local compactness we have $s > 0$; if $s = \infty$ then there is nothing to prove. Assuming that $s < \infty$ we will get a contradiction in two steps. That is we show:

I) $K(z,s)$ is compact

II) There exists a $\delta > 0$ such that $K(z,s+\delta)$ is compact.

Proof of I) : In a complete metrical space, total boundedness of a closed set is equivalent to compactness (see [14], p.87). Therefore we show $K(z,s)$ is totally bounded, i.e., let there be given an arbitrary $\epsilon > 0$ (here $s > \epsilon$), then we must show that $K(z,s)$ can be covered by a finite number of balls where every ball has radius smaller than or equal to ϵ.

To this aim we first consider the ball $K' := K(z, s-10^{-6}\varepsilon)$; here by definition of s, K' is compact and thus totally bounded. Thus, there exists for given $10^{-4}\varepsilon$ a finite covering $(K(z_i, 10^{-4}\varepsilon))$, $1 \leq i \leq n$ for K'. Now for any given $x_s \in K(z,s) \setminus K'$ we have by the above proposition an $x' \in S(z, s-10^{-6}\varepsilon)$ with $d(x', x_s) < d(z, x_s) - (s-10^{-6}\varepsilon) + 10^{-9}\varepsilon$.

Thus $\qquad d(x', x_s) < s - (s-10^{-6}\varepsilon) + 10^{-9}\varepsilon$,

hence $\qquad d(x', x_s) < 2 \cdot 10^{-6}\varepsilon$.

Since $x' \in K'$, it must be contained in some $K(z_m, 10^{-4}\varepsilon)$, $z_m \in \{z_1, .., z_n\}$. This yields with $d(z_m, x_s) \leq d(z_m, x') + d(x', x_s)$ $d(z_m, x_s) < 10^{-4}\varepsilon + 2 \cdot 10^{-6}\varepsilon$, hence $d(z_m, x_s) < \frac{1}{2}\varepsilon$.

We just proved that any given $x_s \in K(z,s) \setminus K'$ has a distance smaller $\frac{1}{2}\varepsilon$ to one of the points z_i , $1 \leq i \leq n$.

Therefore $(K(z_i, \varepsilon))$, $1 \leq i \leq n$, is not only a covering of K' but also of all $K(x_0, s)$; this proves I .

Proof of II) : For any given $x \in M$ we define $\delta(x) := 10^{-9} \sup \{ r \mid K(x,r) \text{ is compact} \}$. By the local compactness for any given $x \in M$ we have $\delta(x) > 0$. (If for one x , $\delta(x) = \infty$, then there is nothing to prove.)

Let now be $(\mathring{K}(x, \delta(x))_x)$ with $x \in K(z,s)$ an open covering of $K(z,s)$, here $\mathring{K}(x, \delta(x)) := \{y \mid d(x,y) < \delta(x), y \in M\}$. Since $K(z,s)$ is compact, we have finitely many points $\{x_1, .., x_l\} \subset K(z,s)$ such that $(\mathring{K}(x_j, \delta(x_j)))$, $x_j \in \{x_1, .., x_l\}$ is a covering of $K(z,s)$. We now define $\delta := \min\{\delta(x_j) \mid 1 \leq j \leq l\}$ and will show in the following that $K(z, s+\delta)$ is compact by proving that $K(z, s+\delta)$ is contained in the compact set

$$\bigcup_{1 \leq j \leq l} K(x_j, 10\,\delta(x_j)) .$$

In order to prove this claim let x^+ be an arbitrary given point in $K(z,s+\delta)\setminus K(z,s)$. Then we get again from our proposition that there is an $\bar{x}\in S(z,s)$ with

$d(\bar{x},x^+) < d(z,x^+) - s + \delta$, thus

$d(\bar{x},x^+) < s + \delta - s + \delta$, hence $d(\bar{x},x^+) < 2\delta$.

Since $\bar{x}\in S(z,s)\subset K(z,s)$, there must exist a point $x_k\in\{x_1,..,x_l\}$ with $\bar{x}\in K(x_k,\delta(x_k))$, and thus $d(x_k,\bar{x}) < \delta(x_k)$.

Combining this, we get by $d(x_k,x^+)\leqq d(x_k,\bar{x})+d(\bar{x},x^+)$

$d(x_k,x^+) < \delta(x_k) + 2\delta \leqq 3\delta(x_k)$.

Thus we have proved that for any given $x^+\in K(z,s+\delta)\setminus K(z,s)$ there exists a certain ball $K(x_k,10\delta(x_k))$ (depending on x^+) which contains x^+. Therefore $\bigcup_{1\leqq j\leqq l} K(x_j,10\delta(x_j))$

does not only contain $K(z,s)$ but also $K(z,s+\delta)$; because every $K(x_j,10\delta(x_j))$ is compact by definition of $\delta(x)$, the statement of II is completely shown.

I,1 Remarks on a counter-example named 'dimension-rambler'

A) After finishing the above proof, some time ago we realized that in [9],p.4/5 a theorem is presented which the author calls Theorem of Hopf and Rinow. Our theorem is included in this result. However the proof there seems to contain a gap. Namely in the proof an argument of the following type is used:

"Let be in a metrical space (M,d) , $K(p,r)=\{x/d(p,x)\leqq r, x\in M\}$ a compact distance ball, and let be $(\mathring{K}(p_i,r_i))$ with $1\leqq i\leqq n$,

* $\mathring{K}(p_i,r_i):=\{x/d(x,p_i)<r_i\}$, $p_i\in K(p,r)$ an arbitrary finite open covering of $K(p,r)$; then there exists some $\delta>0$ such that $K(p,r+\delta)$ is also covered by $(\mathring{K}(p_i,r_i))$, $1\leqq i\leqq n$."

This claim is in general not true, even if (M,d) is a pathwise connected, locally path-connected, locally compact, bounded, complete metrical space. For a counterexample we construct a certain subset which we call dimension-rambler in the Hilbert-space l_2 of all square-summable sequences; for $\bar{x} = (x^i)_{i\in\mathbb{N}} \in l_2$ is defined $|\bar{x}|_2 = (\Sigma |x_i|^2)^{\frac{1}{2}}$. In order to construct the rose-creeper P let $\{\bar{x}_z | z \in \mathbb{Z}\}$ be a set of vectors in l_2 where we define $x_0 = 0$ and for $z \in Z \setminus \{0\}$ we define $\bar{x}_z$ by coordinates in the following way:

$\bar{x}_z^i := \mathrm{sgn}(z)\ (1 + \frac{1}{|z|})\ \delta_i^{|z|}$, where $\delta_i^{|z|}$ denotes the Kronecker-delta and sgn(z) is the signum of z. We connect now the vectors $(\bar{x}_z)_{z\in\mathbb{Z}}$ by an infinitely long polygon. i.e. we define the mapping $P: \mathbb{R} \to l_2$ with

$P(s) := \bar{x}_{[s]} + (s-[s])(\bar{x}_{[s]+1} - \bar{x}_{[s]})$, where $[s]:=\max((-\infty,s)\cap\mathbb{Z})$, (such that for $z\in\mathbb{Z}$ $[z]=z-1$). Considering this polygon piecewise we see that we have segments

$P_z := P([z,z+1]) = \{\bar{x}_z + \lambda(\bar{x}_{z+1} - \bar{x}_z) \mid \lambda \in [0,1]\}$,

every segment $\overline{\bar{x}_z,\bar{x}_{z+1}} := P_z$ connecting two vectors $\bar{x}_z$, $\bar{x}_{z+1}$ $(z\in\mathbb{Z})$.

It can be shown by a straightforward computation that the mapping $P: \mathbb{R} \to (l_2, |\ |_2)$ gives an imbedding of the real line into $(l_2, |\ |_2)$ as a closed subset. The reader should note in this context the fact that

"For arbitrary $z, z' \in \mathbb{Z}$ with $z < z'$, the distance in the

☺ l_2-norm between the subpolygon $P([z,z'])$ and the set $P(\mathbb{R} \setminus [z-1, z'+1])$ is always greater than $\frac{1}{2}$."

This means that, for any two points $P(s')$, $P(s)$ with $|P(s)-P(s')| < \frac{1}{2}$, we must have $|s-s'| < 2$, as can be seen by taking the subpolygon $P([[s],[s]+1])$ and applying ☺.

This last fact is useful by showing that $P(\mathbb{R})$ is closed For our example (M,d) we will take now $M:=P(\mathbb{R})$ endowed with the metric d induced by the l_2-norm on this subspace, i.e. $d(\bar{x},\bar{y}) := |\bar{x}-\bar{y}|_2$ for $\bar{x},\bar{y} \in P(\mathbb{R})$. Clearly $(P(\mathbb{R}),d)$ is a complete metrical space since $P(\mathbb{R})$ is a closed subset of the complete space $(l_2,|\ |_2)$.

By defining $(M,d):=(P(\mathbb{R}),d)$ and taking in (M,d) $K(p,r) := K(0,2^{-\frac{1}{2}})$ together with an arbitrary open covering $\overset{\circ}{K}(p_i,r_i)$, $r_i < \frac{1}{5}$, $p_i \in K(p,r)$ of $K(p,r)$, we claim now that we have a counterexample to (*).

In order to prove this we first show

a) in the space $(P(\mathbb{R}),d)$ the ball $K(0,2^{-\frac{1}{2}})$ equals the compact set $\{\lambda x_1 \mid |\lambda| \leqq \frac{1}{4}\sqrt{2}\}$,

b) for any given $\delta > 0$ we can find some number $z \in \mathbb{Z}$ with $K(0,2^{-\frac{1}{2}}+\delta) \cap P([z,z+1]) \neq \emptyset$.

We will get a) and b) by using the following estimation for $d(0,P(s))$, i.e. with $s = z+\lambda$, $0 < \lambda \leqq 1$, $|z-1| > 1$, we have

$$
\begin{aligned}
d(0,P(s)) &= |P(s)|_2 = |P(z+\lambda)|_2 = |\bar{x}_z + \lambda(\bar{x}_{z+1}-\bar{x}_z)|_2 \\
&= \left(\lambda^2\left(1+\frac{1}{|z+1|}\right)^2 + (1-\lambda)^2\left(1+\frac{1}{|z|}\right)^2\right)^{1/2} \\
&> (\lambda^2+(1-\lambda)^2)^{1/2} \geqq \frac{1}{\sqrt{2}} = 2^{-\frac{1}{2}} .
\end{aligned}
$$

This inequality shows that $P_z \cap K(0,2^{-\frac{1}{2}}) = \emptyset$ for $z \in \mathbb{Z} \setminus \{0,-1\}$ and thus proves a). Further inserting $s=z+\frac{1}{2}$ we get from our estimation that $\left||P(z+\frac{1}{2})|_2 - 2^{-\frac{1}{2}}\right|$ is approaching 0 with $|z| \in \mathbb{N}$ growing; hence the sequence $P(|z|+\frac{1}{2})$, $|z| \in \mathbb{N}$, comes arbitrarily close to the ball $K(0,2^{-\frac{1}{2}})$ and this proves b).

Finally, if we take the open covering chosen above:

$K(p_i,r_i)$, $r_i < \frac{1}{5}$, $p_i \in K(0,2^{-\frac{1}{2}})$ of $K(0,2^{-\frac{1}{2}}) \subset P([-1,1])$, we have $K(p_i,r_i) \cap P_z = \emptyset$ for $|z| > 2$ due to ⊖ .
This finishes the proof of our counterexample.

B) Clearly this example of the metrical space $(P(\mathbb{R}),d)$ also shows that the assumption of an interior metric in the theorem on p. 2 is not superfluous in order to get the metrical space there Heine-Borelsch.

C) Our example $(P(\mathbb{R}),d)$ is also useful in disproving a common error, namely the claim that the distance function attains a minimum measuring the distance between a compact set and a closed set in a complete metrical space (see for instance [1],p.66). For take the compact set $\{0\}$ and the closed set $P(\mathbb{R} \setminus (-1,1))$ in $(P(\mathbb{R}),d)$. Then we have for the distance between $\{0\}$ and $P(\mathbb{R} \setminus (-1,1))$ the value $2^{-\frac{1}{2}}$, but there does not exist any point $q \in P(\mathbb{R} \setminus (-1,1))$ with $d(0,q) = 2^{-\frac{1}{2}}$.
It may be noteworthy that if we modify the definition of (P,d) only very little by defining the coordinate
$\bar{x}_z^i := \operatorname{sgn}(z)\ (2 - \operatorname{sgn}(z) + \frac{1}{z})\ \delta_i^{|z|}$, $z \in \mathbb{Z} \setminus \{0\}$, then we get again the real line embedded as a closed subspace of l_2.
In this case, taking again the distance between $\{0\}$ and $P(\mathbb{R} \setminus (-1,1))$ in the metric induced by the l_2-norm, we get again $\inf\{ d(0,q) \mid q \in P(\mathbb{R} \setminus (-1,1))\} = 2^{-\frac{1}{2}}$, but in this modified case we have in addition that neither this infimum nor the $\sup\{ d(0,q) \mid q \in P(\mathbb{R} \setminus (-1,1))\} = 3$ are attained by points in $P(\mathbb{R} \setminus (-1,1))$. Clearly our modified example of $P(\mathbb{R})$ may also be used for a counterexample against (*) in the same way as the original example was used in A).

§2 Shortest paths in metrical spaces.

We now apply our main result of the first section in order to assure the existence of distance-realizing paths in certain metrical spaces. For this we need the next result which we state without proof, since it is well known and not hard to show, its proof being essentially an application of the Arzela-Ascoli theorem; the reader may find proofs in [2?],p.141 and [8],p.24 .

Theorem: (Hilbert) If in a compact metrical space any two points can be joined by a rectifiable path, then we have a rectifiable shortest connecting path for any two given points in M.

The following result, one of our essential tools, is an immediate consequence of the preceding theorem in combination with our theorem on p.2 which gave sufficient conditions for a metrical space to be Heine-Borelsch.

Theorem: (Hopf , Rinow , Cohn-Vossen) In a locally compact complete metrical space with an interior metric there exists a distance-realizing connecting path for any two given points.

Remark: The condition 'locally compact' in the preceding theorem is not superflous, as can be seen by the following example: take in the Euclidean plane $E^2, |\;|$ a subset $S := \bigcup_{n\in N} (s_n \cup s_n')$, i.e. S is a countable union of sets $(s_n \cup s_n')$ where we define s_n to be the segment connecting the points $(-1,0)$, $(0,\frac{1}{n})$ and s_n' is the segment between $(0,\frac{1}{n})$, $(1,0)$. Now since any two points in S can be joined by a rectifiable path within S, we obviously have an interior metric d in S if we define $d(p,q)$ as the greatest lower bound for the length of paths within S from p to q. Clearly S is complete in its metric since every

Cauchy-sequence has a limit; however there does not exist a path in S from (-1,0) to (1,0) which is distance-realizing. The points (-1,0) , (0,1) have no compact neighbourhood in (S,d). This can be seen as follows: in every ball $K((1,0),\frac{1}{n})$, $n \in N$, in (S,d) we have the subset $A_n := \{x \in S \cap E^2 \mid |(1,0)-x| = (2n)^{-1}\}$, we see that A_n contains a sequence of points in $K((1,0),\frac{1}{n})$ without clusterpoints in S. This is clear because the interior distance between any two different points in A_n equals $\frac{1}{n}$.

§3 Covering spaces and shortest loops.

We want to prove now some results on the existence of rectifiable shortest loops in homotopy classes of loops in a space with an interior metric. For this we use the preceding results together with considerations on the universal covering space of the metrical space. For the general background of the concept of covering spaces we refer to [25],Chap.3 , and [22], Chap.5 .

Let (M,d) be a locally simply connected metrical space with an interior metric. Then since (M,d) is obviously arcwise connected and locally arcwise connected, the universal covering space denoted by $\tilde{M}$ with covering projection $\pi:\tilde{M} \to M$ exists (see for instance [25],p.62). Initially $\tilde{M}$ is a topological space, but since π is a local homeomorphism the metrical structure of (M,d) can be locally lifted to $\tilde{M}$ via π. Thus if $\tilde{U}$ is a neighbourhood of some point $\tilde{p}_0$ in $\tilde{M}$, $\pi(\tilde{p}_0) = p_0$ with $\pi(\tilde{U}) = K(p_0, r)$ where $K(p_0, 2r)$ is evenly covered (i.e. the components of $\pi^{-1}(K(p_0,2r))$ are each mapped homeomorphically onto $K(p_0,2r)$ under π), we define the distance $\tilde{d}(.,.)$ on $\tilde{U}$ by $\tilde{d}(\tilde{p},\tilde{q}) := d(\pi(\tilde{p}),\pi(\tilde{q}))$ for all $\tilde{p},\tilde{q} \in \tilde{U}$. Now paths in $\tilde{M}$ are rectifiable if their projections via π are rectifiable in M.

Since $\tilde{M}$ is connected and because the set of points in $\tilde{M}$ which can be joined by a rectifiable path with some given point is obviously closed and open in $\tilde{M}$, it makes sense to define an interior metric on all $\tilde{M}$, by defining $\tilde{d}(\tilde{p},\tilde{q}) := \inf\{ L(\tilde{c}) \mid \tilde{c}$ is a rectifiable path connecting $\tilde{p}$ and $\tilde{q}$ $\}$. (Of course this is compatible with the preceding local definition, since M has an interior metric.) Further $\tilde{d}$ really defines a metric on $\tilde{M}$: the triangle inequality is trivial to prove so we only check that $\tilde{x} \neq \tilde{y} \in \tilde{M}$ implies $\tilde{d}(\tilde{x},\tilde{y}) > 0$; now if $\pi(\tilde{x}) \neq \pi(\tilde{y})$ we get for every rectifiable connecting path c between $\tilde{x}$ and $\tilde{y}$, $L(\tilde{c}) = L(\pi(\tilde{c})) \geq d(\pi(\tilde{x}),\pi(\tilde{y})) > 0$, since M is a metrical space. In the case $\pi(\tilde{x}) = \pi(\tilde{y})$ by the definition of a covering space we have in M an open neighbourhood, which can be chosen here to be an open ball $K(\pi(\tilde{x}),r)$ with center $\pi(\tilde{x})$ and radius $r > 0$, such that $\pi^{-1}(K(\pi(\tilde{x}),r)$ is a disjoint union of open balls $\tilde{K}(\tilde{x}_i,r)$, $\tilde{x}_i \in \pi^{-1}(\pi(\tilde{x}))$ in $\tilde{M}$; this yields $\tilde{d}(\tilde{x},\tilde{y}) > r > 0$. It is trivial that the topology induced by $\tilde{d}$ on $\tilde{M}$ agrees with the existing topology since the small balls whereon the restriction of π is injective, form a basis of the topology in $\tilde{M}$. By our definition π is a local isometry and we obviously have $\tilde{d}(\tilde{x},\tilde{y}) \geq d(\pi(\tilde{x}),\pi(\tilde{y}))$ for any two given points $\tilde{x},\tilde{y} \in \tilde{M}$. (Of course the above concept of definition for an interior metric in M works in arbitrary covering spaces.)

We will essentially exploit the following

<u>Assertion:</u> Let M be a space with an interior metric and $\pi:\tilde{M} \to M$ be any covering space of M, with $\tilde{M}$ carrying an interior metric d as described above. Then (M,d) is complete if $(\tilde{M},\tilde{d})$ is complete.

<u>Proof:</u> We will only need and only prove that the completeness of (M,d) implies the completeness of $(\tilde{M},\tilde{d})$. For this, let be $(\tilde{x}_n)$ an arbitrary Cauchy-sequence in $\tilde{M}$. Then we get by $d(\pi(\tilde{x}_n),\pi(\tilde{x}_m)) \leqq \tilde{d}(\tilde{x}_n,\tilde{x}_m)$ a Cauchy-sequence $\pi(\tilde{x}_n)$ converging against some $y \in M$. We now choose some $r > 0$ that the balls $\tilde{K}(\tilde{y}_i,r)$, $\tilde{y}_i \in \pi^{-1}(y)$, are all disjoint and $\pi_/ : \tilde{K}(\tilde{y}_i,r) \to K(y,r)$ is isometric. Next we choose $\bar{n} \in \mathbb{N}$ so large that $d(\pi(\tilde{x}_n),y) < \frac{r}{4}$, as well as $\tilde{d}(\tilde{x}_n,\tilde{x}_{\bar{n}}) < \frac{r}{4}$ for all $n \geqq \bar{n}$. Now $\tilde{x}_{\bar{n}} \in \pi^{-1}(K(y,r)) = \bigcup_{\tilde{y}_i \in \pi^{-1}(y)} \tilde{K}(\tilde{y}_i,r)$ must be contained in some ball $\tilde{K}(\tilde{y}_{i_o},r)$, but then all $\tilde{x}_n$ must be in $\tilde{K}(\tilde{y}_{i_o},r)$ for $n \geqq \bar{n}$. This gives by the homeomorphy of $\pi_/ : \tilde{K}(\tilde{y}_i,r) \to K(y,r)$ that $\tilde{x}_n$ is converging to $\tilde{y}_{i_o}$ and proves our claim.

<u>Assertion:</u> Let M be a locally compact, complete, locally simply connected metrical space (M,d) with an interior metric. Then the following statements hold:

a) Let be p,q any two points in M, let be c any path from p to q, then there exists a rectifiable shortest path in the homotopy class of c (p and q may be equal).

b) If M is not simply connected then for any base point p there exists a shortest noncontractible loop $c:[0,l] \to M$, $c(0) = c(l) = p$. If we assume $c(t)$ to be parametrized by arclength then we have $d(p,c(t)) = t$ for $0 \leqq t \leqq \frac{1}{2}l$ and $d(p,c(t)) = l-t$ for $\frac{1}{2}l \leqq t \leqq l$.

c) If M is compact and not simply connected then there exists a shortest noncontractible loop in M.

<u>Proof:</u> For the following $\pi:\tilde{M} \to M$ denotes the above described universal covering of M. In (M,d) any two points can be joined by a distance-realizing path, because M is Heine-Borelsch,

being complete and locally compact, see p. 2 . The completeness of $\tilde{M}$ was proved in the preceding assertion. (For the existence of distance-realizing paths in a Heine-Borelsch metrical space see p. 8.)

a) Take any point $\tilde{p} \in \pi^{-1}(p)$ and lift c beginning in $\tilde{p}$ and connect the endpoint $\tilde{q}$ of the lifted path by a distance-realizing path $\tilde{g}$ with $\tilde{p}$; the projection $(\pi \circ \tilde{g})$ gives a shortest path in the same homotopy class as c. Suppose otherwise, then we would lift a shorter path g' homotopic with c, again beginning the lift in $\tilde{p}$. The lifted path $\tilde{g}'$ would have the same length as g' (π is a local isometry) and would end in $\tilde{q}$, being homotopic to the lift of c ; a contradiction hence, since $\tilde{g}$ was distance-realizing in $\tilde{M}$.

b) Let be $\tilde{p}_o$ any point in $\pi^{-1}(p)$. First $(\pi^{-1}(p) \setminus \{\tilde{p}_o\})$ is closed in $\tilde{M}$, since it has no clusterpoint because there exists $r > 0$ such that $\tilde{d}(\tilde{p},\tilde{p}') \geqq r > 0$ for all $\tilde{p},\tilde{p}' \in \pi^{-1}(p)$, $\tilde{p} \neq \tilde{p}'$. Now the set $A := \tilde{K}(\tilde{p}_o,r') \cap (\pi^{-1}(p) \setminus \{\tilde{p}_o\})$ with $r' = \tilde{d}(\tilde{p}_o,\pi^{-1}(p)\setminus\{\tilde{p}_o\})+1$ is compact being a closed subset of the closed defined and hence by theorem p. 2 compact ball with center $\tilde{p}_o$ and radius r in $\tilde{M}$. Therefore we have a $\tilde{p}_m \in A$ with
$\tilde{d}(\tilde{p}_o,\tilde{p}_m) = \min\{\tilde{d}(\tilde{p}_o,\tilde{p}_i) \mid \tilde{p}_i \in A\} = \tilde{d}(\tilde{p}_o,\pi^{-1}(p)\setminus\{\tilde{p}_o\})$.
We take now a distance-realizing path $\tilde{c}:[0,1] \to \tilde{M}$, $\tilde{c}(0) = \tilde{p}_o$, $\tilde{c}(1) = \tilde{p}_m$, ($\tilde{c}(t)$ in arclength parametrization) , then the projection of $\tilde{c}$, i.e. the loop $\pi \circ \tilde{c} := c$ will fulfil our claim in b). First $\tilde{c}$ is really a shortest noncontractable loop by a similar argument as given in a). For the remaining claim it suffices to show that $d(p,c(\frac{1}{2})) = \frac{1}{2}$, since then any subpath $c:[0,t] \to M$, $t \leqq \frac{1}{2}$, is also distance-realizing, (analogue consideration for

$t \geq \frac{1}{2}$). Suppose otherwise $d(p,c(\frac{1}{2})) < \frac{1}{2}$, then we have a path g from $c(\frac{1}{2})$ to p with length smaller than $\frac{1}{2}$. Now we describe a path denoted by c_1 by first moving along c from $c(0) = p$ to $c(\frac{1}{2})$ and then along g from $c(\frac{1}{2})$ back to p .Lifting c_1 beginning in $\tilde{p}_o$ we get a path $\tilde{c}_1$ in $\tilde{M}$ which must end up in $\tilde{p}_o$ because its length is smaller than 1. But then that part of $\tilde{c}_1$ corresponding to g gives a path of length smaller than $\frac{1}{2}$ from $\tilde{c}(\frac{1}{2})$ to $\tilde{c}(0) = \tilde{p}_o$, this is a contradiction because $\tilde{c}$ was a distance-realizing path in $\tilde{M}$.

c) Let be c_n a sequence of rectifiable noncontractable loops with basepoints p_n in M. Assume that the sequence of their lengths denoted by $L(c_n)$ is monotonically decreasing and converging against the infimum α of the lengths of all rectifiable noncontractable loops in M. Since M is compact we may take now a subsequence of p_n also denoted by p_n which is converging against a point $p_o \in M$. Taking a sequence of distance-realizing paths from p_o to p_n denoted by g_n , we can describe a new sequence of paths $'c_n$ which all have basepoint p_o , i.e. we get $'c_n$ by first walking from p_o along g_n to p_n, then along the loop c_n and finally we return from p_n along g_n back to p_o. We now lift the paths $'c_n$ beginning in some point $\tilde{p}_o \in \pi^{-1}(p_o)$ and denote the lifts by $'\tilde{c}_n$. The sequence $'\tilde{c}_n$ must contain a subsequence whose endpoints $'\tilde{p}_n$ converge against some point $\tilde{p}$ from $\pi^{-1}(p)$ due to the Heine-Borel-property of $\tilde{M}$. We denote this subsequence of $'\tilde{c}_n$ also by $'\tilde{c}_n$. Since $'\tilde{p}_n \in \pi^{-1}(p)$ and because the distance between $\pi^{-1}(p)\setminus\{\tilde{p}\}$ and $\tilde{p}$ is greater than some constant $B > 0$, we get that $'\tilde{p}_n$ must equal $\tilde{p}$ for all n greater than a certain number. Now the lengths of the curves $'\tilde{c}_n$ are converging against α since the lengths of g_n are converging to zero, thus we have $\tilde{d}(\tilde{p}_o,\tilde{p}) \leqq \alpha$. Therefore the projection $(\pi \circ \tilde{c})$ of a distance-realizing path $\tilde{c}$ from $\tilde{p}_o$ to $\tilde{p}$ gives a loop with length $\leqq \alpha$, which is homotopic to $'c_n$. Thus taking $(\pi \circ \tilde{c})$ we have the claimed noncontractible loop with minimal length in M.

Remark: Of course the condition 'locally simply connected' in the preceding theorem is not superfluous in order to assure the existence of a shortest noncontractable loop in a) and b). This can be seen in the following example: take in the Euclidean plane $E^2, |\ |$ a sequence of open balls $D_n := \{ x \mid x \in E^2 , |x-(\frac{1}{n},0)| < \frac{1}{9}(\frac{1}{n}-\frac{1}{n+1}) \}$, $n \in N$. Now we define the following subset M of E^2 , $M := \{ x \in E^2 \mid |x-(0,0)| \leqq 10 \} \setminus D$, $D = \bigcup_{n \in N} D_n$. Clearly M is a compact subset of E^2 for all D_n are open. Since any two points in M can be joined by a rectifiable path in M, we can in a natural way define an interior metric $d(.,.)$ in M, i.e. for $x,y \in M$, $d(x,y) := \inf \{ L(c) \mid c$ is a rectifiable path in M from x to y $\}$. Note here that rectifiability and length L(c) of a path c in M are initially defined respective to the Euclidean metric $d_E(x,y) = |x-y|$ in E^2.(For the definition of length L(c) of a curve c in a metrical space see p. 1 .) However it is easy to see that this length-definition agrees for curves in M with the length definition respective to the just defined metric $d(.,.)$, thus $d(.,.)$ really defines an interior metric on M. Now because we have $1 \leqq d(x,y)/|x-y| \leqq 3$ for all $x,y \in M$, M is compact not only in the Euclidean but also in its interior metric, thus any two points in M can be joined by a distance-realizing path in M. Concerning the original intention of this example, although M is certainly not simply connected, there obviously does not exist any shortest noncontractable loop with basepoint (0,0), due to the fact that M is not locally simply connected at the point (0,0). Further, it is not hard to see that the space in this example contains paths which have no rectifiable homotopic path

1/3' Supplemented remark on

Distance between points in a covering space of a Riemannian Manifold

Let M be a Riemannian manifold with its canonical distance function $d_M: M\times M \to \mathbb{R}$, making M into a metric space (M, d_M).
In a general metric space we can define the length L(c) of a continous curve $c: I \to M$, $I = [a,b]$, by $L(c) = \sup\{c(Z) / Z \in \mathfrak{Z}(I)\}$ with $Z = (a=t_o, \ldots, t_n=b)$ a finite partition of I and $c(Z) := \sum_i d_M(c(t_i), c(t_{i+1}))$; $\mathfrak{Z}(I)$ the set of all finite partitions of I. If L(c) does exist the curve c is called rectifiable.
If for all points $p, q \in M$, $d_M(p,q) = \inf\{L(c) / c$ rectifiable, continous curve from p to q$\}$ the metric space is called a space with an interior metric in the sense of W. Rinow. In a Riemannian manifold the definition of arclength via the integral of the tangent vectors length agrees with the above for piecewise C^1 curves. Therefore a Riemannian manifold is a space with an interior metric in the sense of W. Rinow.

If we have a Riemannian covering $(\tilde{M}, \pi, M)$ of M we can ask:

(*) Is for all pairs of points $p, q \in M$ the distance $d_M(p,q)$ also attained by some pair $\tilde{p}, \tilde{q} \in \tilde{M}$, i.e. $d_{\tilde{M}}(\tilde{p},\tilde{q}) = d_M(p,q)$, $\pi(\tilde{p}) = p$, $\pi(\tilde{q}) = q$? ($d_{\tilde{M}}$ the distance function in $\tilde{M}$)

For a piecewise C^1 curve α connecting p and q in M its lift $\tilde{\alpha}$ via π (for any given initial point $\tilde{p} \in \pi^{-1}(p)$ and uniquely determined endpoint $\tilde{q} \in \pi^{-1}(q)$) has length $L(\tilde{\alpha}) = L(\alpha)$ since _π is a local isometry._
This implies $d_M(p,q) = \inf\{d_{\tilde{M}}(\tilde{p},\tilde{q}) / (\tilde{p},\tilde{q}) \in \pi^{-1}(p) \times \pi^{-1}(q)\}$.
The same does hold if we consider analogue coverings of spaces with interior metric in the sense of Rinow, since here the covering projection is again a _local isometry_ between metric spaces.

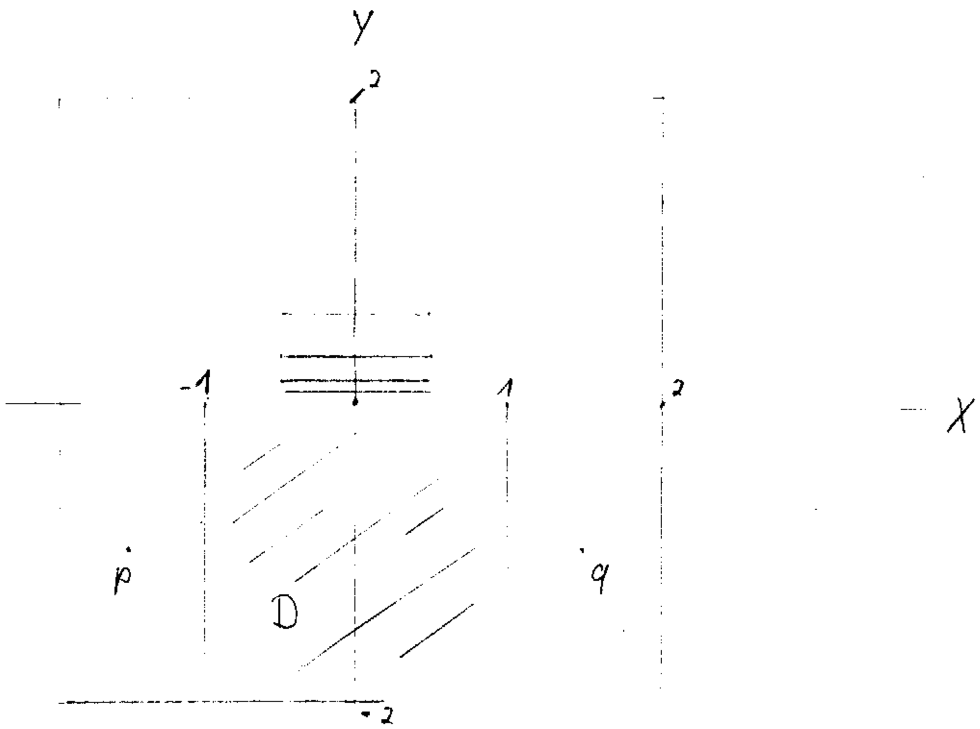
In the Riemannian case isometry has a double meaning. One for the form measuring length of tangentvectors and the other in the sense for ordinary metric spaces. In the Riemannian case π is a local isometry in both senses.

Now if M is a locally compact, complete metric space with an interior metric there exists by a theorem of Rinow a distance realising curve c from p to q , i.e. $L(c) = d_M(p,q)$. A lifted curve $\tilde{c}$ of a rectifiable curve c has the same length as c and therefore the distance $d_{\tilde{M}}(\tilde{p},\tilde{q})$ between the endpoints of the lifted curve must equal $d_M(p,q)$.

In the case of a Riemannian manifold the distance realising curve is a geodesic and its lift too.

This consideration shows that the answer to the question (*) is Yes for M assumed to be complete.

We give now an example of a Riemannian manifold for which the answer to (*) is No .



We define in the Euclidean plane E^2 an open subset $Q = \{ (x,y) \mid \max(|x|,|y|) < 2 \}$ and a subset $D = \{ (x,y) \mid \max(|x|, |y+1|) \leq 1 \}$. Further we define a sequence of segments $S_n = \{ (x,y) \mid y = \frac{1}{n} , |x| \leq \frac{1}{2} \}$.

Let the Riemannian manifold M be now $Q \setminus (D \cup \bigcup_{n \in \mathbb{N}} S_n)$, compare picture. Define now two points $p = (-\frac{3}{2}, -1)$ and $q = (\frac{3}{2}, -1)$. There does obviously not exist a rectifiable curve from p to q within M having length $L(c) = d_M(p,q)$. Now if for a rectifiable curve $c: I \to M$ from p to q $L(c) < d_M(p,q) + \frac{1}{2}$ then we have $c(I) \cap V_n \neq \emptyset$ for exactly one $n_c \in \mathbb{N}$, here $V_n = \{ (y,0) \mid \frac{1}{n} < y < \frac{1}{n-1} \}$, $n \in \mathbb{N}$, $n \geq 2$.

A look at the picture tells that for any curve $c : I \to M$ connecting p and q with length $L(c) < d(p, q) + \frac{1}{2}$ and $c(I) \cap V_{n_c} \neq \emptyset$, c cannot be homotopic to any other rectifiable curve c from p to q with length $L(c) < d(p, q) + \left(\sqrt{\left(\frac{1}{k}\right)^2 + \left(\frac{1}{2}\right)^2} - \frac{1}{2}\right)$ where $k > n_c$.

In order to answer (*) we consider now the universal covering $(\tilde{M}, \pi, M)$ of M. Assume there exist two points $\tilde{p} \in \pi^{-1}(p)$ and $\tilde{q} \in \pi^{-1}(q)$ with $d_{\tilde{M}}(p, q) = d_M(p, q)$. There cannot exist any curve $\tilde{c}$ with $L(\tilde{c}) = d_M(p, q)$ from $\tilde{p}$ to $\tilde{q}$ because then $\pi \circ c$ would be a distance realising curve from p to q, i.e. $L(\pi \circ c) = L(c) = d_M(p, q)$. Assume therefore it exists a sequence $\tilde{c}_m$ of rectifiable curves from $\tilde{p}$ to $\tilde{q}$ having lengths $L(\tilde{c}_m)$ converging against $d_M(p, q)$. All those curves $\tilde{c}_m$ are homotopic since they lie in a simply connected space. But then their projections $\pi \circ c_m$ must all be homotopic curves from p to q whose lengths $L(\pi \circ c_m)$ converge against $d_M(p, q)$. This is a contradiction to our conclusion above.

II INTERIOR METRIC ON MANIFOLDS WHICH SUFFICE A CONE-CONDITION

II/1 Differentiable manifolds with not necessarily smooth boundary fulfiling a cone-condition.

In the following sections we will consider certain differentiable manifolds with boundary where the boundary need not be smooth but only satisfies a so-called cone-condition in every point. For this concept we need some preparations.

A) Notation and assertions for some special types of cones in E^n

For our purposes it will suffice to use very special types of cones. We have to arrange several notations. Let be $K(p,r)$ a closed ball with center p and radius r in E^n, with the origin not contained in $K(p,r)$, thus $|p| > r$. If we take all points lying on the rays issuing from the origin and passing through some point in $K(p,r)$ we get an unbounded closed convex cone denoted by $Co(0,p,r)$. We omit a proof of this obvious fact and refer in this context to [6], p.17. For arbitrary $s > 0$ we define now the following compact, convex cone $C(0,p,r,s) := Co(0,p,r) \cap K(0,s)$; $C(0,p,r,s)$ is compact as a closed subset of the compact ball $K(0,s)$ and convex being the intersection of convex sets. As usually $S(0,|p|)$ denotes the Euclidean distance-sphere with radius $|p|$ around 0. It is obvious that the boundary of $Co(0,p,r)$ is built by the rays corresponding to the set of directional vectors $\{ \frac{x}{|p|} \mid x \in S(p,r) \cap S(0,|p|) \}$. Thus we can describe the boundary of the convex body $C(0,p,r,s)$ by a union of two sets, namely

$$\partial C(0,p,r,s) = \{ t\,x \mid x \in \partial Co(0,p,r),\ |x| = 1,\ 0 \leq t \leq s \} \cup (C(0,p,r,s) \cap S(0,r)),$$

where $\{ t\,x \mid x \in \partial Co(0,p,r),\ |x| = 1,\ 0 \leq t \leq s \}$ equals $\{ \frac{t}{|p|} x \mid x \in S(p,r) \cap S(0,|p|),\ 0 \leq t \leq s \}$.

We omit the proofs of all those facts since they are intuitively clear and can easily be supplied.

Further we know that $C(0,p,r,s)$ as a compact convex body is homeomorphic to the closed n-dimensional unit-disc, thus $\partial C(0,p,r,s)$ is homeomorphic to the (n-1)-dimensional unit-sphere. (See for instance [6], p.21.) We now define the <u>opening-angle</u> of the cone $C(0,p,r,s)$ by the angle between the central axis, here the segment $\{ tp \mid 0 \leq t \leq s |p|^{-1} \}$, and an arbitrary segment issueing from 0 and lying on the boundary of $C(0,p,r,s)$. Clearly the opening-angle denoted by $\alpha = \alpha(0,p,r)$ is well defined and we obviously have $\sin\alpha = \frac{r}{|p|}$, $0 < \alpha < \frac{1}{2}\pi$. Now we will also describe our cones via their opening-angle, i.e. $C(0,\bar{p},\alpha) := C(0,p,r,s)$, with $\sin\alpha = \frac{r}{|p|}$ (remember $r < |p|$) and $\bar{p} = s\frac{p}{|p|}$. This notation expresses the dependence of the three data describing the cone which are the vertex point (here 0), the direction and length of the central axis expressed by the vector $\bar{p}$, and finally the opening-angle α.

<u>Definition:</u> We say that a set G in E^n fullfills an n-dimensional cone-condition or shortly a cone-condition at a point q in G if there exists some cone $C(q,p,\alpha)$ contained in G.

<u>B) Differentiation of functions defined on sets which suffice a cone-condition</u>

Even for sets K in E^n which are not open, it makes sense in certain cases to say that a function $f:K\to\mathbb{R}$ is C^r-smooth, with $1 \leq r \leq \infty$. Assume that K suffices a cone-condition at a point p in K. As usually we call f differentiable at p if there exists a linear mapping $Df(p):E^n\to\mathbb{R}$ with

$$\lim_{|p-q|\to 0} \frac{f(q)-f(p)-Df(p)(q-p)}{|q-p|} = 0 \quad \text{for } q \in K .$$

The linear mapping Df(p) is uniquely determined since there are, for a given basis (v^i) , $1 \leq i \leq n$, in E^n, fixed prescribed values for $Df(p)(v^i)$. This holds because, due to the cone-condition, p can be reached by n linear independent segments $t \to p + t v^i$, $|v^i| = 1$ and $t \in [0, \epsilon)$, contained in K for some $\epsilon > 0$ and because, due to the differentiability of f at p, the right-hand derivatives

$$_{+}v^i_p(f) := \lim_{t \to 0,\, t>0} \frac{f(p+tv^i) - f(p)}{t} \qquad \text{exist ,}$$

prescribing $Df(p)(v^i) = {}_{+}v^i_p(f)$.

Now let us assume that K fullfills a cone-condition at all of its points. Identifying the differential Df(p) and the corresponding gradient vector (via the canonical isomorphism related to the Euclidean scalar product) we say that f is C^1 on K if Df exists on all K and defines a continuous vector field there. Relative to some fixed basis, Df defines on K a mapping with values in $\mathbb{R}^n$. Clearly Df is C^1 if all components of Df are C^1-smooth , and f is called C^2 if Df is C^1-smooth etc. .

Using the material collected in A) and B) we wish to prove now the following proposition basic.

(*)<u>C) Proposition:</u> Let $C(0,p,\alpha)$ be a cone in E^n as described above. Further let $h: C(0,p,\alpha) \to E^n$ be an injective continuous mapping, differentiable in 0 , with derivative $h_0' = L : E^n \to E^n$ a linear isomorphism.

Then the set $h(C(0,p,\alpha))$ fullfills a cone-condition at the point $h(0)$.

<u>Proof:</u> It is obviously no essential restriction if we assume that $h(0) = 0$. First we take a cone $C_{\bar\alpha} := C(0,p,\bar\alpha)$ with $0 < \bar\alpha < \alpha$; thus $C_{\bar\alpha}$ has the same central axis and length as $C_\alpha = C(0,p,\alpha)$, and

(* We can prove that C) Proposition is also valid without the injectivity assumption for the above mapping h.

C_α contains $C_{\bar\alpha}$. Let be y a boundary point of C_α which is not in $S(0,|p|)$, then an elementary geometrical consideration shows that the Euclidean distance between y and $C_{\bar\alpha}$, denoted by $d_E(y,C_{\bar\alpha})$, equals $|y| \sin(\alpha-\bar\alpha)$, since the segment realizing the distance between y and the interior cone $C_{\bar\alpha}$ is normal to the boundary of $C_{\bar\alpha}$ at the point where it touches it. If we define a constant $m(L,\alpha-\bar\alpha) := \sin(\alpha-\bar\alpha) \min_{|x|=1} |L(x)|$ we get for the Euclidean distance between the point $L(y)$ and the set $L(C_{\bar\alpha})$ the following estimate from below:

$d_E(L(y),L(C_{\bar\alpha})) \geqq m(L,\alpha-\bar\alpha)|y|$. (Note that $m(L,\alpha-\bar\alpha) > 0$, since L is an isomorphism.) Now since h is differentiable at 0, we can choose a $\delta_0 > 0$ so small that

$|h_0'(y)-h(0)-h(y)| = |L(y)-h(y)| < 10^{-1} m(L,\alpha-\bar\alpha)|y|$ if $|y| < \delta_0$.

Therefore we get for any given boundary point $h(y)$ of $h(C_\alpha)$ a distance estimation from below to $L(C_{\bar\alpha})$, namely:

$d_E(h(y),L(C_{\bar\alpha})) \geqq \frac{1}{2} m(L,\alpha-\bar\alpha) |y|$ if $|y| < \delta_0 < |p|$.

Note here $|y| < |p|$ assures that p is not in $S(0,|p|)$; the invariance-of-domain theorem assures that every point in $\partial(h(C_\alpha))$ has a preimage in ∂C_α , since h is a homeomorphism onto $h(C_\alpha)$.

For those boundary points of $h(C_\alpha)$ having preimages in $S(0,|p|)$ we can at least say that they stay out of some ball $K(0,m_S)$ for some $m_S > 0$; this holds by a compactness argument. Therefore if we choose $\bar m_S = \frac{1}{2} m_S (\max_{|x|=1} |L(x)|)^{-1}$, then $L(K(0,\bar m_S))$ will be completely contained in $K(0,m_S)$ and thus stay away from those boundary points of C_α which have their preimages in $S(0,|p|)$.

Finally using the just established constants, we define $\widetilde{m} := \min\{\delta_0, \bar{m}_S\}$ and claim that the convex set $L(C(0, \widetilde{m}\frac{p}{|p|}, \bar{\alpha}))$ is contained in $h(C_\alpha)$.

Abbreviating $C(0, \widetilde{m}\frac{p}{|p|}, \bar{\alpha})$ with $\widetilde{C}_{\bar{\alpha}}$, we see that $L(\widetilde{C}_{\bar{\alpha}})$ is a convex body since $\widetilde{C}_{\bar{\alpha}}$ is a convex body (see p. 16.17) and L a linear isomorphism.

In order to prove the inclusion $L(\widetilde{C}_{\bar{\alpha}}) \subset h(C_\alpha)$, let us remark first that above we have chosen $\widetilde{m}$ such that for any point $h(y)$ in $\partial(h(C_\alpha))$ we get $d_E(h(y), L(\widetilde{C}_{\bar{\alpha}})) > 0$ if $h(y) = 0$. This holds because the above estimates imply for $h(y) \in \partial(h(C_\alpha))$ $d_E(h(y), L(\widetilde{C}_{\bar{\alpha}})) \geq \frac{1}{2}\min\{m_S, m(L, \alpha - \bar{\alpha})|y|\}$. That means that the only boundary point of $h(C_\alpha)$ contained in $L(\widetilde{C}_{\bar{\alpha}})$ is 0.

If the convex set $L(\widetilde{C}_{\bar{\alpha}})$ contains an interior point q_I of $h(C_\alpha)$ together with a point q_E of $E^n \setminus h(C_\alpha)$, then by the Jordan-Brouwer separation theorem the segment $\overline{q_I, q_E}$ connecting q_I with q_E necessarily meets the boundary $\partial(h(C_\alpha))$ at some point q_∂ which must be 0 then. (Remember $h(C_\alpha)$ is homeomorphic to the closed n-dimensional unit-disc!) On the other hand, if q_I is in addition an interior point of the convex body $L(\widetilde{C}_{\bar{\alpha}})$, then the segment $\overline{q_I, q_E}$ can only meet $\partial(L(\widetilde{C}_{\bar{\alpha}}))$ at the endpoint q_E. Thus because $0 \in \partial(L(\widetilde{C}_{\bar{\alpha}}))$ we get $q_E = 0$, if $0 \in \overline{q_I, q_E}$. But this is a contradiction since q_E is not in $\partial(h(C_\alpha))$. Therefore the inclusion $L(\widetilde{C}_{\bar{\alpha}}) \subset h(C_\alpha)$ is proved if we show that the convex body $L(\widetilde{C}_{\bar{\alpha}})$ has an interior point which is also interior point of $h(C_\alpha)$. To this aim, note that the distance $d_E(sp, \partial\widetilde{C}_{\bar{\alpha}})$ equals $\min\{|\widetilde{m} - |sp||, |sp|\sin\bar{\alpha}\}$; thus if we take $s > 0$ so small that $|sp|\sin\bar{\alpha} < |\widetilde{m} - |sp||$, then the ball $K(sp, |sp|\sin\bar{\alpha})$ is contained in $\widetilde{C}_{\bar{\alpha}}$. Therefore defining $m := \min\{|L(x)| \mid |x| = 1\}$

we get the following inclusions

$K(L(sp),m|sp|\sin\bar{\alpha}) \subset L(K(sp,|sp|\sin\bar{\alpha})) \subset L(\tilde{C}_{\bar{\alpha}})$.

Using now the differentiability of h at 0, we may choose some $s_o > 0$ so small that $|L(s_o p)-h(s_o p)| < \frac{1}{2} m \sin\bar{\alpha}|s_o p|$, hence the point $h(s_o p)$ being an interior point of $h(C_\alpha)$ is also contained in the interior of $L(\tilde{C}_{\bar{\alpha}})$.
This proves our proposition completely since it is obvious that one can find some cone with vertex 0 in the convex body $L(\tilde{C}_{\bar{\alpha}})$.

We can sharpen now our proposition in the following
<u>Assertion:</u> Let be A a subset of E^n which suffices a cone-condition at a point $q \in A$. Let $h: A \to h(A) \subset E^n$ be a homeomorphism differentiable in q with its derivative a linear isomorphism $h'_q: E^n \to E^n$. Then h(A) fulfilles a cone-condition in h(q) and the inverse of h, $h^{-1}: h(A) \to A$ is differentiable in h(q) with derivative $(h'_q)^{-1}$.
We omit a <u>proof</u> of this assertion since it is not hard to show, now being essentially a corollary of the preceding proposition.

<u>Corollary:</u> Let be A a subset of E^n satisfying a cone-condition at all of its points. Further let $h: A \to h(A) \subset E^n$ be a homeomorphism differentiable with maximal rang at all points in A. Then h(A) satisfies a cone-condition at all of its points and the inverse $h^{-1}: h(A) \to A$ is differentiable on all h(A), with derivative $(h'_q)^{-1}$ at every given point h(q) in h(A).

Although all our preparations are made to treat the case of manifolds having a certain nonsmooth boundary, the concept we present below shall include the cases of smooth boundary and no boundary as well. Therefore we give the following

<u>Definition:</u> An n-dimensional C^r-smooth differentiable manifold M with or without boundary ∂M satisfying a cone-condition:

1) It is a topological n-dimensional Hausdorff-manifold with *) or without boundary, which is connected and has a countable base ;

2) further M carries a C^r-smooth differentiable structure in the following sense:

   A) we can cover M by a family of (in M open) coordinate-neighbourhoods (U_i, φ_i) , every φ_i being a homeomorphism onto $\varphi_i(U_i) \subset E^n$, where

   A') each $\varphi_i(U_i)$ suffices a cone-condition at all of its points, (see p. 18).

   Since obviously (in the subtopology induced by E^n on $\varphi_i(U_i)$) open subsets of $\varphi_i(U_i)$ also satisfy a cone-condition at all of their points, the following requirement makes sense:

   B) for all i, j with $U_i \cap U_j = \emptyset$, $\varphi_i \circ \varphi_j^{-1} / : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j) \subset E^n$ shall be C^r-smooth;

   B') as usual the differentiable structure shall be maximal, i.e. it shall contain all possible compatible coordinate-neighbourhoods.

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*): If boundary occurs, it shall be as usual an (n-1)-dimensional topological submanifold, i.e. we wish that any boundary-point has a neighbourhood homeomorphic to the n-dimensional Euclidean halfspace.

Remark: Since $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is obviously the inverse mapping of $\varphi_i \circ \varphi_j^{-1}$, condition B) assures that $\varphi_i \circ \varphi_j^{-1}$ are diffeomorphisms, thus clearly the differentials of $\varphi_i \circ \varphi_j^{-1}$ have rang n everywhere. It should be said at this place that our proposition on p. 18 obviously assures that the set $\varphi_i \circ \varphi_j^{-1}(\varphi_j(U_i \cap U_j)) \subset E^n$, $U_i \cap U_j \neq \phi$, will automatically satisfy a cone-condition everywhere if $\varphi_i \circ \varphi_j^{-1}$ is injective and differentiable with maximal rang on all $\varphi_j(U_i \cap U_j) \subset E^n$. This means that there exists for the above type of manifolds a large number of fairly natural mappings appropriate to serve as coordinate- exchanges.

Notation: We will often say shortly 'differentiable manifold sufficing a cone-condition' where we always mean an object as described in the preceding definition.

Definition: Let be M,N differentiable manifolds satisfying a cone-condition, as usually a mapping $f: M \to N$ is called C^r-smooth, $(r \geqq 1)$, if $\psi_j \circ f \circ \varphi_i^{-1} : \varphi_i(U_i \cap f^{-1}(V_j)) \longrightarrow \psi_j(V_j)$ is C^r-smooth for all coordinate-neighbourhoods (U_i, φ_i) of M and (V_j, ψ_j) of N for which this condition makes sense, i.e. $V_j \cap f(U_i) \neq \phi$.
Clearly a C^r-smooth diffeomorphism between M and N is C^r-smooth homeomorphism with the inverse mapping also C^r-smooth.

We will now give only scetchy indications in which sense we wish to understand several basic objects related to the manifold M, which are to be used in the sequel.
We restrict ourself to that besause the corresponding concepts developed for manifolds with smooth boundary, as for instance described in ([7], p.140), literally carry over to our case.

We only remark that we always understand by a tangent-space T_pM at a point p in M a full n-dimensional vectorspace, if M is an n-dimensional differentiable manifold satisfying a cone-condition. Note, that due to the cone-condition, n-dimensional vectorspace T_pM roughly speaking can be got as a linear combination of tangent-vectors corresponding to curves which end up in p and <u>are contained</u> in M; thus T_pM is really an object <u>intrinsically</u> related to the manifold M. We denote the tangent-bundle of M by TM. If M is C^r-smooth, $r \geqq 1$, we discribe as usual a C^l-smooth $(r > l \geqq 0)$ Riemannian metric g on M as a C^l-smooth function $g: T^2M \to \mathbb{R}$, where T^2M is as usual the fibre bundle over M with fibre $T_pM \times T_pM$ at a point p in M, and $g_p(.,.) = g : T_pM \times T_pM \to \mathbb{R}$ defines as usual a scalar product on T_pM.

II₂ Interior metric on manifolds which satisfy a cone-condition and carry a Riemannian structure.

Assertion: Let M be an n-dimensional C^r-smooth , $r \geqq 1$, differentiable manifold a cone-condition; further assume that M carries a continuous Riemannian metric g.

Then the following basic facts hold.

1) Any two given points in M can be connected by a continous (piecewise) C^1-smooth path.

2) As usual we define the length $L_D(c)$ of any piecewise C^1-smooth path $c:[a,b] \to M$ by $L_D(c) := {}_a\int^b g(c',c')^{\frac{1}{2}}$.
Further we define the distance $d(p,q)$ between two given points $p,q \in M$ by
$d(p,q) := \inf\{ L_D(c) \mid c$ arbitrary C^1-smooth path from p to q$\}$.
We claim:

A) the function $d: M \times M \to \mathbb{R}$ defines an interior metric on M (see p.1);

B) for any given point $p_o \in M$ we have a coordinate neighbourhood (U,φ) of p_o such that for arbitrary points $p',q' \in U$ the following inequality holds, i.e.
$d(p',q') \geqq |\varphi(p')-\varphi(q')|\, b(\varphi)$, with $|\ |$ the Euclidean norm in $\mathbb{R}^n$ and $b(\varphi) > 0$ some constant depending on the coordinate pair (U,φ) and g ;

C) the topology induced by $d(.,.)$ on M is finer than the manifold topology.

Proof of 1): The manifold M is pathconnected since it is easy to see that a path-connected component of an arbitrary point is nonempty, closed and open in M, thus equals M, because we assumed M to be connected. Further, it is clear that $(M \setminus \partial M)$ is connected because any path in M, connecting two points from $(M \setminus \partial M)$, can obviously be replaced by a path avoiding the boundary. If we consider now for some point $p \in (M \setminus \partial M)$ the subset $C_p^1 \subset (M \setminus \partial M)$ containing only those points which can be joined with p by a (piecewise) C^1-smooth path, it is again clear that C_p^1 is nonempty, open and closed in $(M \setminus \partial M)$ and thus equals $(M \setminus \partial M)$. Therefore in order to connect two arbitrary points in M by a (piecewise) C^1-smooth path, we only need to connect each of them with some interior point of M, but this is obviously possible due to our cone-condition for the manifold.

Proof of 2), A): Using 1) we see that the function $d(.,.)$ is defined on all $M \times M$ with real values greater than or equal to zero. It is trivial that the function $d(.,.)$ is symmetric and that the triangle inequality holds. We only show that for $p,q \in M$ with $p \neq q$ necessarily $d(p,q) > 0$ holds. We know that M is a Hausdorff space with each point having a neighbourhood-basis of compact sets. Thus we may assume that p has a compact neighbourhood U_K with $q \in U_K$ and U_K being contained in some coordinate neighbourhood $(\tilde{U},\varphi)$. Next we take an open neighbourhood U of p contained in U_K. First only using that U is open in M, we see that $\varphi(U)$ is a relative open subset in E^n, since φ is a homeomorphism. This means there exists an in E^n open set O with $\varphi(U) \cap O = \varphi(U)$. If we take now some Euclidean ball $K_E(\varphi(p),r)$, $r > 0$, lying in O, then we get the following relation $(K_E(\varphi(p),r) \cap \varphi(U)) \subset (O \cap \varphi(U)) \subset \varphi(U_K)$.

From this we infer that every continous path $c(t)$, $c:[a,b] \to M$ connecting p with q , i.e. $c(a) = p$, $c(b) = q$, necessarily has a subpath $c:[a,t_m]\to M$, $a<t_m<b$, with $\varphi\circ c([a,t_m])\subset K_E(\varphi(p),r)$ and $|\varphi(p)-\varphi(c(t_m))|=r$. We may define here $t_m := \min\{ t \mid |\varphi(p)-\varphi(c(t))|=r \}$. Now let be c in addition piecewise C^1-smooth, if we can estimate the length of c $c:[a,t_m]\to M$ from below by some constant number $\gamma>0$, γ independent of the specific choice of the path c connecting p with q, then we have $d(p,q)\geqq\gamma>0$ and are finished. For this we describe the length of the tangent vector c' in local coordinates, i.e. $g(c',c')=\langle(\varphi\circ c)',(g_{ik})(\varphi\circ c)'\rangle$, $\langle .,.\rangle$ the Euclidean scalar product and the matrix $g_{ik} := g(\partial/\partial x^i,\partial/\partial x^k)$, where $(x^1,\dots,x^n) := \varphi$. Now since obviously $f(\bar p,\bar x) := \langle\bar x,(g_{ik}(\bar p))(\bar x)\rangle^{\frac12}$ defines a continous function on the compact set $(\varphi(U_K)\times S^{n-1})\subset E^n\times E^n$, $S^{n-1} := \{ x\in E^n \mid |x|=1 \}$, we have $\alpha := \min\{ f(\bar x,\bar p) \mid (\bar p,\bar x)\in(\varphi(U_K)\times S^{n-1})\} > 0$.

Abbreviating $(\varphi\circ c)' =: \bar c'$ we get the estimation

$$\int_a^{t_m}\alpha|\bar c'| = \int_a^{t_m}\alpha(|\bar c'|^2)^{\frac12} \leqq \int_a^{t_m}\langle\frac{\bar c'}{|\bar c'|},(g_{ik})\frac{\bar c'}{|\bar c'|}\rangle^{\frac12}(|\bar c'|^2)^{\frac12} \quad *)$$

$$= \int_a^{t_m}\langle\bar c',(g_{ik})\bar c'\rangle^{\frac12} = \int_a^{t_m} g(c',c')^{\frac12} \quad ,$$

*): at points with $\bar c'=0$ we define the value of the integrand to be zero.

Consequently $L_D(c) \geqq L_D(c/_{[a,t_m]})$

$\geqq \alpha$ (Euclidean arclength of $\bar c/_{[a,t_m]}$)

$\geqq \alpha|\bar c(a)-\bar c(t_m)| = \alpha r$,

thus we have $d(p,q)\geqq\alpha r>0$. This shows that $d(.,.)$ is a metric on M.

It remains to explain why $d(.,.)$ gives an interior metric on M. (See definition p. 1). Since we know that (M,d) is a metrical space we can define for a continous curve $c:[a,b]\to M$ the so-called '<u>rectifiable length</u>'

$L_R(c) := \sup\{\sum_i d(c(t_i),c(t_{i+1})) \mid z:=(..t_i \leq t_{i+1}..)$ arbitrary finite partition of $[a,b]\}$.

In order to show that $d(.,.)$ gives an interior metric, it remains to check that always $L_R(c) \geq d(p',q')$ for any path connecting any two points $p',q' \in M$. But this is an immediate consequence of the triangle inequality which yields here $\sum_i d(c(t_i),c(t_{i+1})) \geq d(c(a),c(b)) = d(p',q')$ for an arbitrary partition $(..t_i \leq t_{i+1}..)$ of $[a,b]$, thus $L_R(c) \geq d(p',q')$.

<u>Proof of 2), B):</u> Using the estimate developed above it is now clear that for any two points p',q' with $\varphi(p')$ and $\varphi(q')$ elements of $K_E(\varphi(p),\frac{1}{2}r)$ we have $d(p',q') \leq \min\{\alpha|\varphi(p')-\varphi(q')|, \alpha r\} \leq \alpha|\varphi(p')-\varphi(q')|$, thus in the neighbourhood $\varphi^{-1}(K_E(\varphi(p),\frac{1}{2}r) \cap \varphi(U))$ of p we have the claim of 2)B) valid with $b(\varphi) = \alpha$.

<u>Proof of 2),C):</u> If a sequence (p_n) is converging against some point p in the metrical space (M,d), then the estimations established above show that for sufficiently large $n \in \mathbb{N}$ the sequence $|\varphi(p_n)-\varphi(p)|$ is defined and converges to zero. Thus the sequence p_n is converging against p in the manifold-topology and this gives the continuity of the identity $id:(M,d) \to (M,\text{manifold topology})$ and proves 2)C).

<u>Remark:</u> Although it was possible to define via a Riemannian structure an interior metric on manifolds which suffice a cone-condition, this class of manifolds still allows too much pathologies for our purposes. For instance, the topology induced by (M,d) on M need not to agree with the manifold topology. It is not so hard to give examples for this fact, even in two di-

mensions. Even if the boundary curve is C^∞ everywhere except at one point, it is possible to have a sequence p_n converging against one point p in the manifolds topology, while the interior distance $d(p_n,p)$ is unbounded.

We will restrict our considerations from now on to differentiable manifolds a cone-condition and having the additional property that every point has a neighbourhood diffeomorphic to a convex set in E^n.

II 3 Interior metric on manifolds which are locally diffeomorphic to a convex set in E^n.

Assertion: Let M be an n-dimensional C^r-smooth differentiable manifold with or without boundary a cone-condition and let every point have a neighbourhood C^l , $r \geqq l \geqq 1$, diffeomorphic to a convex set in E^n. We assume further that M carries a continous Riemannian metric g.

Then the following statements hold:

A) The distance $d(.,.)$ defined via the Riemannian metric g gives a space with an interior metric (M,d). (See p. [illegible].25 for definitions.)

B) Every neighbourhood of any given point p contains compact and open neighbourhoods of p which are C^1-diffeomorphic to convex sets with interior points in E^n.

C_1) For every neighbourhood $\tilde{U}$ of any given point p_0 we have a coordinate neighbourhood (U,φ) of p_0 , $U \subset \tilde{U}$, with two constants $b(\varphi)$, $B(\varphi) > 0$, such that
$b(\varphi) \leqq d(p,q)/|\varphi(p)-\varphi(q)| \leqq B(\varphi)$ holds for all $p,q \in U$.
Moreover this neighbourhood U may be chosen as a distance-ball $K(p,r') = \{ q \in M \mid d(p,q) < r' \}$.

C_2) The topology induced on M by the interior metric $d(.,.)$ coincides with the manifolds topology.

C_3) Let (M_i,d_i), $i \in \{1,2\}$, be metrical spaces, let further $f_1:(M_1,d_1) \to (M,d)$, $f_2:(M,d) \to (M_2,d_2)$ be two continuous mappings. Then

a) f_1 is locally Lipschitz-continous iff there exists for any given $x \in M_1$ a distance-ball $K_1(x,r) = \{ x' \in M_1 \mid d_1(x,x') < r \}$ and a coordinate neighbourhood $(K(f_1(x),r'),\varphi)$ in M such that $f_1(K(x,r)) \subset K(f_1(x),r')$ and $(\varphi \circ f_1)$ is (locally) Lipschitz-continous;

b) f_2 is locally Lipschitz-continous iff for any $p \in M$ there exists a neighbourhood (U,φ) such that $(f_2 \circ \varphi^{-1}): \varphi \ (U) \to (M_2,d_2)$ is locally Lipschitz-continous.

Proof of A): This has been shown in the assertion p.25 .

Proof of B): By assumption we have a coordinate pair (U,φ) with φ C^1-smooth and with a set $C \subset U$, C neighbourhood of p and $\varphi(C)$ convex in E^n. Next we take neighbourhoods A and O of p with A compact, O open and $O \subset A \subset C \subset U$. Now $\varphi(O)$ is relative open in $\varphi(U)$ hence there is a set O_E open in E^n with $O_E \cap \varphi(U) = \varphi(O) = O_E \cap \varphi(O) \ni \varphi(p)$. We take further a Euclidean ball $K_E(\varphi(p),r) = \{ x \in E^n \mid |\varphi(p)-x| \leq r \}$ contained in O_E. We claim $V_E := \bar{K}_E(\varphi(p),r) \cap \varphi(U)$ is a compact convex set with interior points. In addition, defining $\mathring{V}_E := \mathring{K}_E(\varphi(p),r) \cap \varphi(U)$, $\mathring{K}_E(\varphi(p),r) := \{ x \in E^n \mid |\varphi(p)-x| < r \}$, we claim $\mathring{V}_E$ is convex with interior points in E^n. Under those assumptions $\varphi^{-1}(V_E)$ is a neighbourhood of p , C^1-diffeomorphic to a compact, convex body in E^n, and $\varphi^{-1}(\mathring{V}_E)$ is a neighbourhood C^1-diffeomorphic to a convex set with interior points in E^n. It remains to show that V_E is a compact, convex body in E^n. For this goal, using $K_E(\varphi(p),r) \subset O_E$ we get $V_E = (K_E(\varphi(p),r) \cap \varphi(U)) \subset (O_E \cap \varphi(U)) = \varphi(O)$ thus $V_E \subset (K_E(\varphi(p),r) \cap \varphi(O))$, hence by $O \subset U$ we get $V_E = K_E(\varphi(p),r) \cap \varphi(O)$. Therefore $O \subset A \subset C \subset U$ yields $V_E = K_E(\varphi(p),r) \cap \varphi(C) = K_E(\varphi(p),r) \cap \varphi(A)$. Thus V_E is compact, convex being intersection of convex sets (i) and compact sets (ii); (i) also shows the convexity of the relative open set $\overset{\circ}{V}_E$, clearly $\varphi^{-1}(\overset{\circ}{V}_E)$ is an open neighbourhood of p contained in $\varphi^{-1}(V_E)$. Finally $\overset{\circ}{V}_E \subset E^n$, $\overset{\circ}{V}_E$ being homeomorphic to an open set in the n-dimensional Euclidean halfspace must contain interior points by the invariance of domain theorem.

This proves B).

<u>Proof of C_1):</u> In an analogue way as in the proof of 2),A) on p. 2[illegible] we estimate (from below and above) the distance between two points $p', q' \in \varphi^{-1}(\tilde{V}_E)$, $\tilde{V}_E := K_E(\varphi(p), \frac{1}{2}r) \cap \varphi(O)$, V_E and $\varphi(O)$ as in B). Thus defining $b(\varphi) := \min\{ f(\bar{p},\bar{x}) \mid (\bar{p},\bar{x}) \in V_E \times S^{n-1} \}$, $B(\varphi) := \max\{ f(\bar{p},\bar{x}) \mid (\bar{p},\bar{x}) \in \tilde{V}_E \times S^{n-1} \}$ with $f(\bar{p},\bar{x})$ as explained on p. 2[illegible] , we get the claim of C_1) for the neighbourhood $\varphi^{-1}(\tilde{V}_E)$ of the point p. <u>Note</u>, <u>here</u> we use for the estimation $d(p',q') \leq B(\varphi)|\varphi(p')-\varphi(q')|$, $p',q' \in \varphi^{-1}(\tilde{V}_E)$, that there exists a <u>Euclidean</u> segment in $\tilde{V}_E$ <u>connecting $\varphi(p')$ with $\varphi(q')$ due to the convexity of $\tilde{V}_E$</u>.

<u>Proof of C_2):</u> The statement of C_1) clearly proves C_2) , since the convergence of a sequence p_n against a point p in the manifolds topology means that the sequence $|\varphi(p_n)-\varphi(p)|$ (explained for n large enough) converges to zero and thus by C_1) forces $d(p,p_n)$ to be a zero sequence. Now for the remaining claim of C_2) we define $r' = b(\varphi)\frac{1}{2}r$ and have $\varphi(K_E(p,r')) \subset \tilde{V}_E$. (We know

that $K_E(p,r')$ is a neighbourhood of p also in the manifolds topology.) Now the last inclusion assures that the estimations * are valid for the neighbourhood $K_E(p,r')$ of p. This means that the restriction $\varphi : (K_E(p,r'),d) \to (\varphi(K_E(p,r')), |\ |)$ is a Lipschitz-continous homeomorphism with Lipschitz-continous inverse with Lipschitz constants $b(\varphi)$, $B(\varphi)$.
Here $K_E(p,r')$ shall carry the metric induced by $d(.,.)$ and $\varphi(K_E(p,r'))$ the induced Euclidean metric.

Proof of C_3): Since the composition of Lipschitz-continous mappings is again Lipschitz-continous, the last statement immediately implies all C_3).

The existence of enough examples of manifolds of the type in the last assertion even with the additional property of metrical completeness is guaranteed by the following

Assertion and corollary: Let M be an n-dimensional C^1-smooth manifold without boundary, with continous Riemannian metric g. Let M be complete in its interior metric $d(.,.)$. If we have a topological n-dimensional submanifold S with boundary such that every point on ∂S has a neighbourhood C^1-diffeomorphic to a convex set in E^n, then S itself, as manifold in the sense of the last assertion, is complete in its own interior metric iff it is complete as a metrical subspace of (M,d). (For the interior distance on S, $d_S : S \times S \to \mathbb{R}$, we only consider paths which are contained in S.)

The proof is an immediate application of the preceding assertion, especially of C_1) and C_2) .

III SMOOTHNESS PROPERTIES OF SHORTEST PATHS

III 1 The length of an absolutely continuous path

We want to make some preparations now, for the subsequent smoothness investigations of shortest rectifiable paths in Riemannian manifolds with boundary. To this purpose we prove a result which compares the different definitions for arclength of a curve in a Riemannian manifold. This result, the following lemma, seems to be of interest by itself.

Lemma: Let be (M, g, d) an n-dimensional manifold as described in the Assertion on p. 22. This means every point in M has a neighbourhood C^1-diffeomorphic to a compact convex set in R^n. As on p. 22 g denotes here a continuous Riemannian metric and d(.,.) the corresponding canonical distance function on M. Then for an arbitrary absolutely*) continous path $c: I=[a,b] \to (M,d)$ we have $L_I(c) = \int_I g(c',c')^{1/2} = L_R(c)$, where $L_R(c)$ denotes the rectifiable length of c related to the distance d in M. (See page 1, 22, 24 for a definition.)

Proof: It is sufficient to prove the statement for some given part of c which is contained in any arbitrarily small neighbourhood U of some given point q of c in M. (We denote this piece of c also with c.) The following estimation procedures can be performed more easily in a vector space. Therefore we consider a C^1-smooth isometric embedding

**) $f: (U,g) \to E^1$ of some small compact, in R^n convex subset U which carries the Riemannian structure induced by (M,g). Here the restriction of g on U is also denoted by g. E^1 is a Euclidean vectorspace of sufficiently high dimension. That such a C^1-smooth isometric imbedding exists at least for a sufficiently small open neighbourhood around some given point in a C^∞-smooth differentiable manifold which carries a continuous Riemannian metric is not so difficult to prove, and is assured for instance by a more general theo-

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**): We abbreviate here in the use of our notation, more precisely we mean by (U,g) a representation in local coordinates.

*): Note by the Assertion p. 29, C_1 any path $c: I \to (M,d)$ is absolutely continuous in the metrical space (M,d) iff for any coordinate pair (φ, U) with $\varphi(U)$ compact, convex $\varphi \circ c$ is absolutely continuous in E^n. This holds due to the fact that Lipschitz-continuous mappings preserve the absolute continuity of the path.

rem of J.Nash (see [23], p.383,). In order to get such an open neighbourhood of q appropriate in our situation, we first construct a Riemannian prolongation of (U,g) to an open set $\tilde{U}$ which contains U and on which we define a metric $\tilde{g}$. For this purpose we take some fixed interior point m in U. For each single ray issueing from m we define the metric $\tilde{g}$ at the points of the ray outside of U by taking the metric g at the intersection point of this ray with the boundary of U. On U we define $\tilde{g}$ by g. Using the convexity of U we see that our construction of $\tilde{g}$ is well defined and that $\tilde{g}$ is continous on $\tilde{U}$ if g was continous on U. Some in $\tilde{U}$ open neighbourhood $\tilde{O}$ of q, carrying the metric *) induced by $\tilde{g}$ can now be isometrically imbedded into some E^l by the theorem above. Next we consider an in R^n compact, convex neighbourhood of q contained in $\tilde{O}$. The intersection of this neighbourhood with U is again compact, convex and contains q. Restricting the isometric imbedding to this subset carrying the metric induced by g, finally yields an isometric imbedding of a neighbourhood of q. But this is now a neighbourhood of q relative to the topology in U and thus to the topology in M. Without changing notation we denote this neighbourhood of q in U, together with its metric again by (U,g).

The lemma is proved now in two steps. In the first step we prove the inequality

(I) $$L_{R_E}(c) \leq L_{D_E}(c) \quad ,$$

in the second step the inequality

(II) $$L_{R_U}(c) \leq L_{R_E}(c) \quad ,$$

here $L_{R_U}(c)$, $L_{R_E}(c)$ denote the rectifiable length of c in (U,g) and the Euclidean space E^l respectively,

*): Since $\tilde{O}$ is an open set in E^n we may interprete the pair $(\tilde{O}, \tilde{g})$ as an C^∞-manifold carrying a continuous Riemannian metric. Thus Nash's theorem [23] can literally be applied.

$L_{D_E}(c)$, $L_{D_U}(c)$ denote the differentiable length (see page P4) in the related structures. Because c in (U,g) is isometrically imbedded together with (U,g) in E^l, we get $L_{D_E}(c) = L_{D_U}(c)$.

The interior distance in (U,g) defined by those paths which lie completely in U defines a distance-function which agrees with the distance-function of M at least on a sufficiently small open neighbourhood of q in U. Thus at least within this neighbourhood we can say $L_R(c) = L_{R_U}(c)$.

Combining now both inequalities we finally get the claim of the Lemma, i.e. $L_D(c) = L_R(c)$.

<u>Proof of inequality (I)</u> : Let $c:[a,b] \to E^l$ be an absolutely continous curve; to prove (I) it suffices to show

$$d_E\,(c(a),c(b)) := |\,c(a)-c(b)\,| := \Big(\sum_{i=1}^{l} |\,c_i(a)-c_i(b)\,|\Big)^{\frac{1}{2}}$$

$$(A^l) \qquad \leq \int_a^b \Big(\sum_{i=1}^{l} |\,c_i'\,|^2\Big)^{\frac{1}{2}}$$

At first the existence of the right-hand-side of the inequality (A^l) can be seen as follows: c_i' is in $L^1([a,b])$ for $1 \leq i \leq l$, thus $\infty > \sum_{i=1}^{l} \int_a^b |\,c_i'\,| = \int_a^b \sum_{i=1}^{l} |\,c_i'\,|$

$$\geq \int_a^b \Big(\sum_{i=1}^{l} |\,c_i'\,|^2\Big)^{\frac{1}{2}} .$$

The proof of (A^l) goes now by induction over the number of components of c'. The inequality (A^l) is valid with l=1 :

$$(A^1) \quad (\,|\,c_1(a)-c_1(b)\,|^2\,)^{\frac{1}{2}} \leq \int_a^b |\,c_1'\,| \quad \text{is true because}$$

$c_1(b)-c_1(a) = \int_a^b c_1'$ yields

$$|\,c_1(b)-c_1(a)\,| = \Big|\int_a^b c_1'\Big| \leq \int_a^b |\,c_1'\,| .$$

Using the abbreviation $|\,x\,|_k := \Big(\sum_{i=1}^{k} x_i^2\Big)^{\frac{1}{2}}$, $1 \leq k \leq l$, x an element of E^l, the induction hypothesis (A^{l-1}) reads as

$$(A^{l-1}) \qquad |\,c(a)-c(b)\,|_{l-1} \leq \int_a^b |\,c'\,|_{l-1} .$$

From (A^{l-1}) and (A^{1}) we get $|c(a)-c(b)|_{l}$ $=$

$$(|c(a)-c(b)|_{l-1}^{2} + |c_{l}(a)-c_{l}(b)|^{2})^{\frac{1}{2}} \leq$$

$$\left(\left(\int_a^b |c'|_{l-1}\right)^{2} + \left(\int_a^b |c_{l}'|\right)^{2} \right)^{\frac{1}{2}} .$$

We now apply a wellknown inequality from the theory of L^{p}-spaces: for $k>1$ and $F,G \in L^{1}([a,b],R)$ we have

$$\left[\left(\int |F|\right)^{k} + \left(\int |G|\right)^{k} \right]^{1/k} \leq \int (|F|^{k} + |G|^{k})^{1/k} ,$$

see for instance [18], p.77 . If we apply this inequality to $F = |c'|_{l-1}$, $G = |c_{l}'|$ and $k=2$ and use the last inequality above, we get

$$|c(a)-c(b)|_{l} \leq \int_a^b (|c'|_{l-1}^{2} + |c_{l}'|^{2})^{\frac{1}{2}} .$$

For any vector $c'(t)$ in E^{l} we have the identity

$$|c'(t)|_{l-1}^{2} + |c_{l}'(t)|^{2} = |c'(t)|_{l}^{2}$$. Using this,

we finally obtain inequality (I) in the notation

$$|c(a)-c(b)|_{l} \leq \int_a^b |c'|_{l} .$$

Proof of inequality (II) : for any given $\epsilon > 0$ we show the existence of a $\delta(\epsilon)$ with

$$S_{M}(c,Z(\delta)) \leq S_{E}(c,Z(\delta)) + \epsilon \quad \text{for all } \delta < \delta(\epsilon) .$$

Here denotes $Z(\delta) = (a=t_{1}<..<t_{i}<..<t_{n}=b)$ an arbitrary partition of I with $0<t_{i+1}-t_{i}\leq\delta$ for $1\leq i\leq n-1$, further

$$S_{M}(c,Z(\delta)) := \sum_{i} d(c(t_{i}),c(t_{i+1})) \quad \text{and}$$

$$S_{E}(c,Z(\delta)) := \sum_{i} |f(c(t_{i}))-f(c(t_{i+1}))| .$$

We have f' uniformly continous on U, thus there is a $\delta_f(\epsilon)$ with $|f'(x)-f'(y)| < \frac{1}{2}\epsilon(L_{R_E}(c))^{-1}$ for all $x,y \in U$ with $|x-y| < \delta_f$ (here the Euclidean norms in R^n and R^1 are both denoted by $|.|$). Further, since f is C^1-smooth, the remainder term

$$R(x,x-y) := \frac{f(x)-f(y)-f'(x)(x-y)}{|x-y|}$$ is a uniformly

continous function $R(x,h)$ (with $y=:x+h$) on a compact set $U \times K(\theta,r)$ in R^{2n}, where $K(\theta,r)$ is an appropriate chosen compact ball of radius r around the origin $\theta = 0$ (see for instance p.62). Therefore it exists $\tilde{\delta}_f(\epsilon)$ such that $|R(x,x-y)| < \frac{1}{2}\epsilon(L_{R_E}(c))^{-1}$ if $|x-y| = |h| < \tilde{\delta}_f(\epsilon)$.

We finally choose our $\delta(\epsilon)$ required above so small that $|c(t)-c(\hat{t})| < \min\{\delta_f(\epsilon),\tilde{\delta}_f(\epsilon)\}$ for $|t-\hat{t}| < \delta(\epsilon)$.

Note that for the fixed given mapping f and curve c, $\delta(\epsilon)$ is indeed only depending on ϵ.

For an arbitrary given from now on fixed partition $Z(\delta)$ of I, $\delta < \delta(\epsilon)$, we define a piecewise C^1-smooth curve $P:I \to U$ by $P(t) := c(t_i) + \frac{t-t_i}{t_{i+1}-t_i}(c(t_{i+1})-c(t_i))$ for $t_i \leq t \leq t_{i+1}$, $t_i \in Z(\delta)$; the convexity of U assures that P(I) is contained in U. We get

$$S_M(c,Z(\delta)) \leq \sum_i \int_{t_i}^{t_{i+1}} |(f\circ P)'(t)| =$$

$$= \sum_i \int_{t_i}^{t_{i+1}} \left| f'(P(t)) \frac{c(t_{i+1})-c(t_i)}{t_{i+1}-t_i} \right| = \sum_i |f'(P(\alpha_i))\,\Delta(t_i,c)|$$

with $t_i < \alpha_i < t_{i+1}$ and $\Delta(t_i,c) := c(t_{i+1})-c(t_i)$.

Applying now the estimations established above, we have

$$\begin{aligned}S_M(c,Z(\delta)) &\leq \sum_i \left(|f'(c(t_i))\Delta(t_i,c)| + \tfrac{1}{2}\epsilon\,(L_{R_E}(c))^{-1}|\Delta(t_i,c)|\right)\\ &\leq \sum_i |f'(c(t_i))\Delta(t_i,c)| + \tfrac{1}{2}\epsilon\\ &\leq \sum_i \left(|f'(c(t_i))\Delta(t_i,c)| - |R(c(t_i),\Delta(t_i,c))|\,|\Delta(t_i,c)|\right) + 2\tfrac{1}{2}\epsilon\\ &\leq \sum_i |f'(c(t_i))\Delta(t_i,c) + R(c(t_i),\Delta(t_i,c))|\Delta(t_i,c)|\,| + \epsilon\\ &= \sum_i |f(c(t_{i+1}))-f(c(t_i))| + \epsilon\\ &= S_E(c,Z(\delta)) + \epsilon\ .\end{aligned}$$

So inequality (II) is verified and the Lemma completely proved.

II/2 Smoothness of a rectifiable shortest path

(1) Theorem: Let M be a C^2-smooth manifold as described in the Assertion p. 29, let every point have a neighbourhood C^2-smooth diffeomorphic to a convex set in E^n. This means every point in M has a neighbourhood C^2-diffeomorphic to an n-dimensional compact convex set in R^n (see p.29). Further we assume that M has a locally Lipschitz-continous Riemannian metric. Let be $c:[\bar{a},\bar{b}] \to M$ an arbitrary (rectifiable) shortest path in arclength parametrization.
If then $[a,b]$ is any subinterval of $[\bar{a},\bar{b}]$ not containing a and b, then the restriction of c on $[a,b]$ has an absolutely continous first derivative and and square Lebesgue-integrable second derivative on $[a,b]$.

Proof: We can cover the path $c([\bar{a},\bar{b}])$ by a finite number of charts (U_i,φ_i), $1\leq i\leq m$, with $\varphi_i(U_i)$ compact convex sets in R^n. Further we can assume that there exists h_o with $0 < h_o \leq \min\{|a-\bar{a}|,|b-\bar{b}|\}$ such that every U_i contains a subpath $c([a_i-h_o,b_i+h_o]$, $1\leq i\leq m$, with the following conditions for the subpaths: $a_1=a$, $b_m=b$, $a_i<b_i$ for $1\leq i\leq m$, $a_{i+1}<b_i$ for $1\leq i\leq m-1$; then clearly $\bigcup_{1\leq i\leq m}[a_i,b_i] = [a,b]$.
We will restrict our considerations now to one subpath $\varphi_j\circ c :[a_j-h_o,b_j+h_o] \to \varphi_j(U_j) =: O_j$; we denote $\varphi_j\circ c$ also by c. We are finished if we can prove that for arbitrary given $j\in\{1,..,m\}$ $\varphi_j\circ c :[a_j,b_j] \to \varphi_j(U_j)$ has absolutely continous first derivative and square Lebesgue-integrable second derivative on $[a_j,b_j]$. The proof will consist in showing that we have a constant B_j that gives an upper bound for ${}_j|\Delta_h c|_1 = (\int_{a_j}^{b_j}|\Delta_h c|^2 + |(\Delta_h c)'|^2)^{\frac{1}{2}}$ for all $h < h_o/2$, here $(\Delta_h c)(t) := \dfrac{c(t+h)-c(t)}{h}$.

Since c is Lipschitz-continous we get our claim then by theorem p. 59 in the appendix. (Note initially c is Lipschitz continuous in the metrical space (M,d), but by p. 29, 31 G_1 we have here also $\varphi_j \circ c$ Lipschitz continuous respective the Eucledean metric in $E^n \supset O_j$.)

In order to show the existence of a bound for ${}_j|\Delta_h c|_1$ we will procede now in several steps.

I. In the first step we prove that for any Lipschitz-continous curve $\bar{c}:[x,y] \to M$, $[x,y] \subset [\bar{a},\bar{b}]$, with $\bar{c}(x) = c(x)$ and $\bar{c}(y) = c(y)$, we have $\int_x^y \tilde{g}(c') \leq \int_x^y \tilde{g}(\bar{c}')$, where $\tilde{g}(c') = g(c',c')$ is the quadratic form related to the Riemannian scalar-product (more detailed written $g(c',c') = g_{(c(t))}(c',c')$, $t \in [x,y]$).

II. In what follows we will consider $c := \varphi_j \circ c \to O_j$ on the interval $[a_j - h_0, b_j + h_0] =: [e,d]$. We introduce a specific test-function $c - \epsilon\Phi_h$, with $\Phi_h := -\Delta_{-h}\{\eta^2 \Delta_h c\}$, $h < h_0/4$, where η is a C^∞-smooth cut-off function with $\eta \equiv 1$ on $[a_j, b_j]$ and $\eta \equiv 0$ on $[e, e+\frac{1}{2}h_0]$ and $[d-\frac{1}{2}h_0, d]$ (η is only depending on h_0).

We will show in II that for any such given Φ_h we have an $\epsilon(\Phi_h)$, i.e. an ϵ depending on Φ_h, such that for all $0 \leq \epsilon < \epsilon(\Phi_h)$ $(c - \epsilon\Phi_h)([e,d]) \subset O_j$. This will mean that Φ_h is an admissible *) testvector for any given $h < h_0/4$, since it is Lipschitz-continous. The Lipschitz-continuity holds because multiplication of a Lipschitz-continous function with a Lipschitz-continous function again gives a Lipschitz-continous function, and because $\Delta_h w$ is Lipschitz-continous if the function w was so.

*) Admissible means here that the energy of $c - \epsilon\Phi_h$ can be compared with the energy of c by I, since I only makes sense if $c - \epsilon\Phi_h$ stays in M.

III. In the third step, knowing that $c-\epsilon\Phi_h$ is admissible, we use the inequality I with $\bar{c} := c-\epsilon\Phi_h : [e,d] \to O_j$:

$$0 \leq {}_e\int^d (\tilde{g}(\bar{c}') - \tilde{g}(c')) = {}_e\int^d (\tilde{g}(c'-\epsilon\Phi_h') - \tilde{g}(c')) .$$

Taking the limit $\lim\limits_{\epsilon\to 0+} \frac{1}{\epsilon}\{{}_e\int^d (\tilde{g}(c'-\epsilon\Phi_h')-\tilde{g}(c')) \}$

we will get the inequality

$$0 \leq -2\,{}_e\int^d \Phi_h'^{\,t}\, \bar{g}(c)\, c' + 2\,{}_e\int^d |c'|^2 M\, |\Phi_h| \quad ;$$

(for the definition of $\bar{g}(c)$, see p.42).

IV. In the last step we will finally use the inequality derived in III to show that ${}_j|\Delta_h c|_1$ is bounded by some constant B_j for all $0 < h \leq h_o/4$.

Proof of I : In order to prove I, remember that c is a Lipschitz-continous curve and the manifold M has a boundary such that we can apply lemma p.33 , which says that the rectifiable length $L_R(c)$ and the differentiable length $L_D(c)$ agree; further because c is an arclength parametrized shortest curve between the endpoints, we have

$$L_R(c) = {}_x\int^y \tilde{g}(c')^{\frac{1}{2}} = {}_x\int^y \tilde{g}(c') = |y-x| \leq L_R(\bar{c}) = L_D(\bar{c})$$
$$= {}_x\int^y \tilde{g}(\bar{c}')^{\frac{1}{2}}$$

because $\bar{c}$ also is a Lipschitz-continous curve from $\bar{c}(x)$ to $\bar{c}(y)$. Therefore

$${}_x\int^y \tilde{g}(c')^{\frac{1}{2}} \leq \frac{L_D(\bar{c})}{|y-x|} L_D(\bar{c}) \leq {}_x\int^y \tilde{g}(\bar{c}') \quad , \text{ by using}$$

the Schwarz-inequality $({}_x\int^y fk)^2 \leq {}_x\int^y f^2\, {}_x\int^y k^2$ with $f \equiv 1$ and $k = \tilde{g}(c')^{\frac{1}{2}}$. This gives

$${}_x\int^y \tilde{g}(\bar{c}') \leq {}_x\int^y \tilde{g}(\bar{c}') \quad .$$

Proof of II : In order to prove $(c-\epsilon\Phi_h)([e,d])$ is contained in the convex set O_j, we simply show that $(c-\epsilon\Phi_h)([e,d])$ can be got by convex combinations of points from $c([e,d])$ and thus will stay in the convex set O_j. Namely

$$(c-\epsilon\Phi_h)(t) = c(t)-\epsilon\Phi_h(t) = c(t)+\epsilon\Delta_{-h}\{\eta^2(t)\Delta_h c(t)\}$$
$$= \lambda_1 c(t+h) + \lambda_2 c(t-h) + (1-\lambda_1-\lambda_2)c(t) \quad ,$$

where $\lambda_1 = \frac{\epsilon}{h^2}\eta^2(t)$, $\lambda_2 = \frac{\epsilon}{h^2}\eta^2(t-h)$; hence , for $0 < \epsilon < \frac{h^2}{2}$, we have that $0 \leq \lambda_1,\lambda_2 \leq \frac{1}{2}$.
Thus we infer that for all $t\in[e,d]$, $(c(t)-\epsilon\Phi_h(t))$ is a convex combination of the three points $c(t+h)$, $c(t)$, $c(t-h)$. Here for $(t-h)\leq e$ we define $\eta(t-h) = 0$ and $c(t-h) = c(e)$; analogous for $(t+h)\geq d$; see also [16], p.17/18.

Proof of III : We know that $0 \leq \int_e^d\{\tilde{g}(c'-\epsilon\Phi_h')-\tilde{g}(c')\}$.
If we denote by $\bar{g}(c)$ the matrix related to the symmetric bilinear form $g_c(.,.)$, relative to the local coordinates in O_j, then we can describe $\tilde{g}(c')$ by $c'^t\,\bar{g}(c(.))\,c'$. The notation $c'^t\bar{g}(c)c'$ indicates that the bilinear form is depending on the footpoint of the vector c'. With this notation the above inequality reads as

$$0 \leq \int_e^d\{(c'-\epsilon\Phi_h')^t\,\bar{g}(c-\epsilon\Phi_h)\,(c'-\epsilon\Phi_h') - c'^t\bar{g}(c)c'\} \quad , \text{ i.e.}$$
$$0 \leq \int_e^d\{-2\epsilon\Phi_h'^t\,\bar{g}(c-\epsilon\Phi_h)c' + \epsilon^2\Phi_h'^t\,\bar{g}(c-\epsilon\Phi_h)\Phi_h' +$$
$$+ c'^t(\bar{g}(c-\epsilon\Phi_h)-\bar{g}(c))c'\} \; .$$

(We use the symmetry of the matrix.) Now because $\bar{g}(.)$ is Lipschitz-continous, we have $|\bar{g}(c-\epsilon\Phi_h)-\bar{g}(c)| \leq M|\epsilon\Phi_h|$, which implies

$$_e\int^d c'^t(\bar{g}(c-\epsilon\Phi_h)-\bar{g}(c))c' \leqq {}_e\int^d c'^t(M\epsilon|\Phi_h|)c' = {}_e\int^d |c'|^2 M\epsilon|\Phi_h| .$$

Therefore we get

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\ {}_e\int^d\{-\epsilon 2\Phi_h'^t\,\bar{g}(c-\epsilon\Phi_h)c' + \epsilon^2\Phi_h'^t\,\bar{g}(c-\epsilon\Phi_h)\Phi_h' + {}+ |c'|^2 M\epsilon|\Phi_h|\}$$

$$= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\epsilon\ {}_e\int^d\{-2\Phi_h'^t\,\bar{g}(c-\epsilon\Phi_h)c' + \epsilon\Phi_h'^t\,\bar{g}(c-\epsilon\Phi_h)\Phi_h' + {}+ |c'|^2 M|\Phi_h|\} .$$

Because we started above with $0 \leqq \epsilon\ {}_e\int^d(\ldots)$, the integral $_e\int^d(\ldots)$ is greater or equal to 0 for all $\epsilon>0$. From this we get

$$0 \leqq {}_e\int^d -2\Phi_h'^t\,\bar{g}(c)c' + {}_e\int^d 2|c'|^2 M|\Phi_h| .$$

This holds because

$$|\,{}_e\int^d -2\Phi_h'^t\,\bar{g}(c)c' - {}_e\int^d\{-2\Phi_h'^t\,\bar{g}(c-\epsilon\Phi_h)c' + \epsilon\Phi_h'^t\,\bar{g}(c-\epsilon\Phi_h)\Phi_h'\}\,|$$

$$= |\,{}_e\int^d -2\Phi_h'^t\,(\bar{g}(c)-\bar{g}(c-\epsilon\Phi_h))c' + {}_e\int^d \epsilon\Phi_h'^t\,\bar{g}(c-\epsilon\Phi_h)\Phi_h'\,|$$

$$\leqq {}_e\int^d 2\,|\Phi_h'|\,|c'|\,\epsilon M|\Phi_h| + {}_e\int^d \epsilon\,|\Phi_h'|^2 \max_{x\in O_j}|\bar{g}(x)|$$

$$\leqq {}_e\int^d |c'|^2 M\,|\Phi_h| \quad \text{for all } \epsilon>0 \text{ sufficiently small.}$$

Proof of IV : In the last part of our proof we will use rules from the calculus of difference-quotients, which we explain in detail in the appendix, see p. 56 to p. 61 . Choosing another constant $\bar{M} \geqq M|c'|^2$, we write the inequality got in III as

$$0 \leqq -{}_e\int^d\Phi_h'^t\,\bar{g}(c)c' + {}_e\int^d\bar{M}\,|\Phi_h| , \qquad \text{i.e.}$$

$$_e\int^d\Phi_h'^t\,\bar{g}(c)c' \leqq {}_e\int^d\bar{M}\,|\Phi_h| .$$

Using ([illegible]) p. 61 , (V1) p. 56 and the condition that $0 < |h| < \frac{1}{4}|h_o|$ and $\eta \equiv 0$ on $[e, e+\frac{1}{4}h_o]$, $[d-\frac{1}{4}h_o, d]$:

$$_e\int^d D(-\Delta_{-h}\{\eta^2\Delta_h c\})^t\,\bar{g}(c)c' \leqq {}_e\int^d \bar{M}\,|-\Delta_{-h}\{\eta^2\Delta_h c\}|\quad,$$

$$_e\int^d -\Delta_{-h}\{D(\eta^2\Delta_h c)\}^t\,\bar{g}(c)c' \leqq {}_e\int^d \bar{M}\,|D(\eta^2\Delta_h c)|\quad.$$

Applying now (R_1) p. 61 and the product-rules (R_{2iv}) p. 61 , we get

$$_e\int^d (D(\eta^2\Delta_h c))^t\Delta_h\{\bar{g}(c)c'\} \leqq {}_e\int^d \bar{M}\,|D\eta^2(\Delta_h c)+\eta^2 D\Delta_h c|\,.$$

Next we apply (R_{2iv}) p. 61 and introduce the shifted vector $c'_{(h)}$ by $c'_{(h)}(t) := (Dc)(t+h)$ to find out that

$$_e\int^d \{D(\eta^2)(\Delta_h c)+\eta^2 D\Delta_h c\}^t\{(\Delta_h\bar{g}(c))c'_{(h)}+\bar{g}(c)\Delta_h Dc\} \leqq$$

$$\leqq {}_e\int^d \bar{M}\,\eta^2\,|D\Delta_h c| + B$$, where B is some constant real number with $_e\int^d \bar{M}\,|D\eta^2(\Delta_h c)| \leqq B$. This gives

$$_e\int^d \eta^2\,(\Delta_h Dc)^t\,\bar{g}(c)\,\Delta_h Dc \leqq$$

$$_e\int^d |\Delta_h Dc|\,\{\bar{M}\eta^2+\eta^2|\Delta_h\bar{g}(c)|\,|c'_{(h)}|+|\bar{g}(c)|\,|\Delta_h c|\,|D\eta^2|\} + E + B\,,$$

where $E := {}_e\int^d |D\eta^2(\Delta_h c)|\,|\Delta_h\bar{g}(c)|\,|c'_{(h)}|$. In order to prepare for applying the Schwarz-inequality we derive from the above inequality the following one:

$$_e\int^d \eta^2|\Delta_h Dc|^2 \min_c\{\min_{|x|=1}|\bar{g}(c)x|\} \leqq$$

$$_e\int^d \eta\,|\Delta_h Dc|\,\{\bar{M}\eta+\eta|\Delta_h\bar{g}(c)|\,|c'_{(h)}|+|\bar{g}(c)|\,|\Delta_h c|\,|2D\eta|\} + E + B\,.$$

Abbreviating $\min_c\{\min_{|x|=1}|\bar{g}(c)x|\} =: K$ (clearly $K>0$) , $\sup_{c(t)}\{\bar{M}\eta+\eta\,|\Delta_h\bar{g}(c)|\,|c'_{(h)}| + |\bar{g}(c)|\,|\Delta_h c|\,|2D\eta|\} =: P$ and applying the Schwarz-inequality, we get

$$_e\int^d (\eta|\Delta_h Dc|)^2 \leqq {}_e\int^d \eta\,|\Delta_h Dc|\frac{P}{K} + \frac{1}{K}(E+B) \leqq$$

$$\leqq \left({}_e\int^d \eta^2|\Delta_h Dc|^2\right)^{\frac{1}{2}}\left({}_e\int^d (P/K)^2\right)^{\frac{1}{2}} + \frac{1}{K}(E+B)\,.$$

This yields

$$\left({}_e\!\int^d (\eta^2 |\Delta_h Dc|^2) \right)^{\frac{1}{2}} \leqq \left({}_e\!\int^d (P/K)^2 \right)^{\frac{1}{2}} + \frac{1}{K}(E+B) \Big/ \left({}_e\!\int^d \eta^2 |\Delta_h Dc| \right)^{\frac{1}{2}} .$$

Thus we finally have

$$\left({}_e\!\int^d (\eta^2 |\Delta_h Dc|^2) \right)^{\frac{1}{2}} \leqq \left({}_e\!\int^d (P/K)^2 \right)^{\frac{1}{2}} + \frac{1}{K}(E+B) + 1 \quad .$$

We know that η^2 , c , $\bar{g}(c)$ are Lipschitz-continous functions, hence their difference-quotients and derivatives are bounded, therefore the right-hand side in the last inequality is bounded. So we finally conclude that

$$_j|\Delta_h c|_1^2 = {}_{a_j}\!\int^{b_j} |\Delta_h c|^2 + {}_{a_j}\!\int^{b_j} |\Delta_h Dc|^2$$

stays bounded by some number, constant for all h with $0 < h < \frac{1}{4}h_o$, because $\eta \equiv 1$ on $[a_j, b_j] \subset [e,d]$.
This shows that we have a common bound for all $_j|\Delta_h c|_1$ and finishes the proof of our theorem.

<u>Remark:</u> It is possible to sharpen the preceding theorem, i.e. it can be shown that under the same condition as given in theorem (I), a rectifiable shortest path is everywhere C^1-smooth with absolutely continous derivative and square integrable second derivative. However the proof of this statement does not simply follow from the preceding theorem, it needs an additional argument, it is possible to give a proof using the theorem above in combination with a geometrical consideration.

III$_3$ Applications

We will finally present several applications of our theorem, which we get partly in combination with the existence results from the first part of this paper which was dealing exclusively with metrical spaces without the additional assumption of a smooth structure.

*) For the following, if there is said nothing different, M shall always be an n-dimensional C^2-smooth differentiable manifold satisfying a cone-condition, being locally C^2-diffeomorphic to a convex set in R^n. Further M shall carry a locally Lipschitz-continous Riemannian metric. M carries by the assertion on p. 21 a canonical interior metric; we denote the related distance by $d(.,.)$, in the covering space $\tilde{M}$ respectively by $\tilde{d}(.,.)$. We always assume arclength parametrization for the considered curves.

(II) <u>Theorem:</u> Let M be as described in *), then the following statements hold:

A) Let p,q be any two points in M and let be $c:[0,a] \to M$, $c(0) = p$, $c(a) = q$, a rectifiable shortest path in a homotopy class of paths from p to q (p and q may be equal). Then c is C^1-smooth with absolutely continous derivative and square integrable second derivative on every compact subinterval of (0,a).

B) Let $c:[0,a] \to M$ be any rectifiable, on all M shortest, non-contractible loop, i.e. $c(0) = c(a)$.
Then c is C^1-smooth with absolutely continous derivative and square integrable second derivative on all [0,a].

Remark: If we assume M only to be C^1-smooth, and locally C^1-smooth diffeomorphic to a convex set in R^n, but having the local representation of the Riemannian metric $(g_{ik}):\varphi(U) \to R^n$ locally Lipschitz-continuous for every coordinate neighbourhood (U,φ) ; then again A) and B) hold, but only with the C^1-smooth property of the paths there. This is evident if one remembers that the regularity considerations were locally performed in the Euclidean chartspace E^n related to a coordinate neighbourhood (U,φ). Thus we get again by the same local arguments as in the proof of our theorem I p. 39 that we have a C^1-curve with absolutely continous derivative etc. in the chartspace E^n, but this gives in the manifold structure only a C^1-smooth curve. (The same is valid for the sharpened version of theorem I on p. 39 as it is described without proof in the above remark.)

Proof of A): We take the universal covering $\tilde{M}$ with covering projection $\pi:\tilde{M} \to M$. Clearly $\tilde{M}$ exists since M suffices all necessary connectedness assumptions, see. [25], p.62; π is C^2-smooth and a locally isometric (local) diffeomorphism; M is also a space with an interior metric $\tilde{d}(.,.)$, see p. 9/10. Now we take any point $\tilde{p} \in \pi^{-1}(p)$ and lift the path c, thus we get a path $\tilde{c}:[0,a] \to \tilde{M}$ with $\pi \circ \tilde{c} = c$, $\tilde{c}(a) = \tilde{q} \in \pi^{-1}(q)$. The path $\tilde{c}$ is a rectifiable shortest connection between the points $\tilde{p}, \tilde{q}$ in $\tilde{M}$, since otherwise there would exist a strictly shorter rectifiable path $\tilde{c}^{+}$ between $\tilde{p}$ and $\tilde{q}$ in $\tilde{M}$, whose projection $\pi(\tilde{c}^{+})$ would give a rectifiable loop in the same homotopy class as c but shorter than c; a contradiction hence. Therefore we have our claim by applying now theorem (I), p. 39 to $\tilde{c}$ in $\tilde{M}$ and then getting c by the C^2-smooth projection π from $\tilde{c}$.

<u>Proof of B):</u> Clearly by A) c is C^1-smooth etc. on every compact subinterval of $(0,a)$. If we take now a path $\bar{c}(s) := c(s+\varepsilon):[\varepsilon, a+\varepsilon] \to M$, $0 < \varepsilon$ small, we obtain a path of equal length and in the same free homotopy class as c, thus also noncontractable. This gives by A) again that $\bar{c}$ must be C^1-smooth etc. on every compact subpath of $(\varepsilon, a+\varepsilon)$ and proves B).

The following corollary is obtained by combining the preceding theorem with the existence theorem on p. 8 and p. 11.

(III) <u>Corollary:</u> a) Let M be as described in *) on p. 46, further let M be complete in its interior metric $d(.,.)$; then we have for any two points $p,q \in M$ a distance-realizing path $c:[0,d(p,q)] \to M$, where c is C^1-smooth, with absolutely continous derivative and square integrable second derivative on every compact subinterval of $(0,d(p,q))$.

a') The existence of a path as described in A) in the preceding theorem is assured if (M,d) is complete. (In the simply connected case the shortest loop is a constant mapping onto the base point.)

$\bar{a}$) If M is complete and not simply connected, then for every $p \in M$ we have a homotopy class of loops which contains a with respect to the given basepoint p shortest noncontractable loop $c:[0,a] \to M$, $c(0) = c(a) = p$ (by assertion b) p.11) which is by (IIA) everywhere C^1-smooth except eventually in the basepoint p.
b) The existence of a path as described in B) in the preceding theorem is assured if M is compact (and not simply connected).

<u>Assertion:</u> Let M be as described in *) on p.46 , let us assume in addition the Riemannian metric g to be C^1-smooth and the boundary of M to be empty. Then any shortest rectifiable connecting path $c:[0,d(p,q)] \to M$ between any two points $p,q \in M$ is a geodesic on all $[0,d(p,q)]$.

<u>Proof:</u> Since c is C^1-smooth with absolutely continous derivative on every compact subinterval I of $(0,d(p,q))$, we get by the same variational procedure as for C^2-curves that c suffices the differential equations of geodesics almost everywhere on I, i.e. in local coordinates $c^{r''}(t) = -\Gamma^r_{ik}(c(t))\, c^{i'}(t)\, c^{k'}(t)$ holds for almost all $t \in I$. Using the continuity of $\Gamma^r_{ik}(c(t))$ and the Lipschitz-continuity of $c^i(t)$, we get some constant B independent of the chosen subinterval I such that $|c^{r''}(t)| < B$ holds almost everywhere on I. Exhausting $[0,d(p,q)]$ by a countable sequence of subintervals, we have $|c^{r''}(t)| < B$ almost everywhere on $[0,d(p,q)]$. This soon yields the existence of some constant $\tilde{B}$ with $\tilde{B} > |c'|_1 \geq |\Delta_h c'|_0$ by using V_1 on p.56, <u>here</u> for the norms $|\;|_1$ and $|\;|_0$ (see p.52 for definitions) the integral is taken over <u>every compact intervall in $(0,d(p,q))$</u> . Applying now the regularity theorem B) p.53 , we have that $c^{r'}(t)$ is absolutely continous

on all $[0,d(p,q)]$, thus it can be represented as an integral of its derivative $c^{r''}(t)$ which coincides almost everywhere with the continous function $-\Gamma^r_{ik}(c(t))\, c^{i'}(t)\, c^{k'}(t)$. Therefore $c^{r'}(t)$ equals an integral of a continous function and thus is necessarily C^1-smooth on all $[0,d(p,q)]$. This proves that c is a C^2-smooth curve sufficing the equations for geodesics on all $[0,d(p,q)]$ and hence is a geodesic as we claimed.

(IV) Corollary:

1) (Hopf and Rinow): In a C^2-smooth metrically complete Riemannian manifold we have for any two points a distance-realizing geodesic connection.

2) (Hadamard , Berger): If M is a compact C^2-smooth Riemannian manifold then we have a shortest closed noncontractible geodesic , if M is not simply connected.

Proof: The combination of corollary (III) with the preceding assertion proves this statement.

(V) Corollary: Let M be a manifold as described in *) on p.46 , let M be metrically complete. If there exists one point $p \in M$ such that the squared distance-function $d^2(p,.)$ is directionally differentiable on all M, then M is necessarily simply connected.

Proof: Assume otherwise, then we have by (III, $\bar{a}$) a shortest noncontractable loop $c:[0,a] \to M$, $c(0)=c(a)=p$, with c C^1-smooth on $(0,a)$, $c(t)$ given in arclength parametrization; using b) in assertion p.11 we have $d(p,c(t))=t$ for $0 \leq t \leq \frac{a}{2}$, and $d(p,c(t))=a-t$ for $\frac{a}{2} \leq t \leq a$. Since the function $t \to d(p,c(t))$ is not differentiable at $t=\frac{a}{2}$, we see that the directional derivative of the function $d(p,.)$ at the point $c(\frac{a}{2})$ in the direction of the tangentvector of the curve c there, i.e. for $c'(\frac{a}{2})$ cannot exist.

Remark: A) If M is simply connected, then also $M \setminus \partial M$ is simply connected, $(M \setminus \partial M)$ being a weak deformation retract of M (see [26],p.297), hence being of the same homotopy type as M (see [26],p.33). We know that an open simply connected twodimensional topological manifold is homeomorphic to the open twodimensional unit disc (see [?]). Therefore it can be shown that if the manifold M in the preceding corollary is twodimensional, then under the conditions of the corollary M is homeomorphic to the (open twodimensional unit disc plus an open subset B of the unit circle), where each connected component of B corresponds to a boundary component of M.

This result may be interesting in relation with the following one which R. Bishop proved in [4] and we proved in [28] i.e.:

If M is an n-dimensional open (i.e. $\partial M = \phi$) complete Riemannian manifold having at least one point p with the square distance function $d^2(p,.)$ directionally differentiable on all M, then M is diffeomorphic to the open n-dimensional unit disc.

In this theorem the Riemannian metric is assumed to be C^2-smooth.

Appendix A: Analytic tools

We present in the sequel several basic facts from Analysis in order to make the whole paper more selfcontained.

The spaces H_0 and H_1 and imbedding results.

The imbedding results in this paragraph are special cases of theorems mainly mainly due to Rellich and Sobolev.

Let E^n be an n-dimensional Euclidean vectorspace with Euclidean scalar-product $\langle .,. \rangle$ and corresponding norm $| . |$. Given any closed interval $[a,b]$, we consider the following spaces: first the Hilbert-space $H_0([a,b],E^n)$, short H_0 of square (Lebesgue-) integrable curves of $[a,b]$ in E^n, i.e. for f in H_0 we have $\int_a^b \langle f,f \rangle < \infty$. (More precisely an element in H_0 is a class of mappings where the members need only to be defined almost everywhere on $[a,b]$, none the less with an abuse of language we always speak about square-integrable curves where we really mean certain classes of almost everywhere defined measurable functions; two members of one class coincide almost everywhere on $[a,b]$.) The scalar-product $\langle .,. \rangle_0$ on H_0 is defined by $\langle f,g \rangle_0 := \int_a^b \langle f,g \rangle$, $f,g \in H_0$, and the norm $| \; |_0$ by $|f|_0^2 = \langle f,f \rangle_0$.

Further we consider the spaces $H_1([a,b],E^n)$, short H_1, of absolutely continous curves from $[a,b]$ into E^n with derivative contained in H_0.

It is a basic fact from real-analysis that an absolutely continous function $f:[a,b] \to E^n$ is almost everywhere differentiable on $[a,b]$, its derivative f' is measurable and $\int_a^b |f'| < \infty$, further f can be represented as an integral of its derivative, i.e. $f(t) = \int_a^t f'(s)ds + \text{const.}$, $t \in [a,b]$, see [ ], p.283.

The space H_1 with the scalar-product $\langle f,g \rangle_1 = \langle f,g \rangle_0 + \langle f',g' \rangle_0$ and $|f|_1^2 := \langle f,f \rangle_1$ is in fact a real Hilbert-space, see [ ], p. .

Since the polynomials are dense in $(H_0, | \, |_0)$ (even those with rational coefficients), it is not difficult to see, that the polynomials are also dense in $(H_1, | \, |_1)$.

So both H_0 and H_1 can also be got as a completion of the pre-Hilbert-space of C^∞-smooth functions over $[a,b]$ (or even of the polynom-space), relative to the norms $|\,|_0$, $|\,|_1$.
We also use the Banach-space of continous functions $C^0([a,b],E^n)$ with the norm $|f|_\infty := \sup\{|f(t)|\ ,\ t\in[a,b]\}$.
Further we employ the following facts:

Assertion: The inclusions

$$H_1([a,b],E^n) \to C^0([a,b],E^n) \to H_0([a,b],E^n)$$

are continous , more precisely:

(i) if $f\in C^0$, then $|f|_0 \leq |f|_\infty\, |a-b|^{\frac{1}{2}}$,

(ii) if $g\in H_1$, then $|g|_\infty \leq |g|_1\ 2\,(|a-b|^{\frac{1}{2}}+|a-b|)$.

(iii) Moreover the inclusions $H_1 \to C^0$ and $H_1 \to H_0$ are compact operators; we even have that bounded, closed balls from H_1 are mapped onto compact sets by those inclusion-operators.

We first prove the following useful inequality for a curve f in H_1 (see also [13], p.178).
For an arbitrary finite set $s_k < t_k \leq s_{k+1} < t_{k+1}$ $(k=1..N)$ $a \leq s_1$, $t_{N+1} \leq b$, we have

(iv) $$\sum_k |f(s_k)-f(t_k)| \leq \Big(\sum_k |s_k-t_k|\Big)^{\frac{1}{2}} \langle f',f'\rangle_0^{\frac{1}{2}}$$

$$\leq \Big(\sum_k |s_k-t_k|\Big)^{\frac{1}{2}} |f|_1 \qquad .$$ Namely

$$\Big(\sum_k |f(s_k)-f(t_k)|\Big)^2 \leq \Big(\sum_k \int_{s_k}^{t_k} |f'(t)|dt\Big)^2 =$$

$$= \Big(\int_{U_N} |f'(t)|\ 1\ dt\Big)^2 \leq \int_{U_N} 1\,dt \int_{U_N} |f'|^2 ,$$

where $U_N := \cup\,[s_k,t_k]$. To get the last inequality, we may interprete the left side as an integral over $[a,b]$ where the integrand equals zero on $[a,b]\setminus U_N$ and apply the Schwarz - inequality with this integral over $[a,b]$. Now we have $$\Big(\sum_k |f(s_k)-f(t_k)|\Big)^2 \leq \int_{U_N} 1\ dt \int_{U_N} |f'|^2$$

$$\leq \Big(\sum_k |f(s_k)-f(t_k)|\Big) \langle f',f'\rangle_0$$

which proves (iv).

(iv) explicitly reproves that f is absolutely continous. Specializing (iv) with N=1 , we get

$|f(s)-f(t)| \leq |s-t|^{\frac{1}{2}}\langle f',f'\rangle_0^{\frac{1}{2}} \leq |s-t|^{\frac{1}{2}}|f|_1$, what means that f is Hölder - $\frac{1}{2}$ - continous.

Proof of (ii): we choose $t_0 \in [a,b]$ with

$|g(t_0)| = \min\{|g(t)| , t\in[a,b]\}$; then

$|g(t_0)||a-b| \leq \int_a^b |g(t)|$, which implies

$$
\begin{aligned}
|g(t)| &\leq |g(t_0)| + |g(t_0)-g(t)| \\
&\overset{(iv)}{\leq} \left(\int_a^b |g(t)|\right)|a-b|^{-1} + |a-b|\,\langle g',g'\rangle_0^{\frac{1}{2}} \\
&\leq |a-b|^{\frac{1}{2}}\,\langle g,g\rangle_0^{\frac{1}{2}}\,|a-b|^{-1} + \langle g',g'\rangle_0^{\frac{1}{2}}|a-b| \\
&\leq (|a-b|^{-\frac{1}{2}}+|a-b|)\,(\langle g,g\rangle_0^{\frac{1}{2}} + \langle g',g'\rangle_0^{\frac{1}{2}}) \\
&\leq (|a-b|^{-\frac{1}{2}}+|a-b|)\,2\,|g|_1 .
\end{aligned}
$$

At the end we used that both $\langle g,g\rangle_0^{\frac{1}{2}}$, $\langle g',g'\rangle_0^{\frac{1}{2}} \leq |g|_1$.

Proof of (i) : we have $\int_a^b |f(t)|^2 \leq |f|_\infty^2\,|a-b|$, thus

$$\langle f,f\rangle_0^{\frac{1}{2}} \leq |f|_\infty\,|a-b|^{\frac{1}{2}} .$$

Proof of (iii) : we want to show first that

$(H_1,|.|_1) \to (C^0,|.|_\infty)$ is a compact operator (synonym to completely-continous operator). For this we show that bounded sets are mapped onto sequentially relative-compact sets. (In a metrical space, sequential compactness and compactness agree.) The proof is an application of the Arzelà-Ascoli-Theorem, since a bounded subset in H_1 is a uniformly bounded, equicontinous family of functions, where equicontinuity of the family is assured by the Hölder-$\frac{1}{2}$-continuity in (iv).

Hence any in H_1 bounded set contains a sequence converging in the norm $|.|_\infty$ against a continous function. Since we have established (i), the same sequence is also converging in $(H_o,|.|_o)$ against a continous function. Therefore $(H_1,|.|_1) \rightarrow (C^o,|.|_\infty)$ and $(H_1,|.|_1) \rightarrow (H_o,|.|_o)$ are indeed completely continous operators.

Now in order to prove that bounded closed balls from $(H_1,|.|_1)$ are mapped <u>onto</u> compact sets, it is clearly sufficient to show this for the closed unit-ball $K := \{ x , x \in H_1 \text{ and } |x|_1 \leq 1 \}$. For this purpose we view at H_1 endowed with its related weak topology denoted (H_1,W). Then the inclusions $(H_1,W) \rightarrow (C^o,|.|_\infty)$, $(H_1,W) \rightarrow (H_o,||_o)$ are sequentially continous, since the corresponding operators viewed over $(H_1,||_1)$ are completely-continous operators between normed spaces (see [24], p.177). Now K is sequentially compact in (H_1,W) because $(H_1,||_1)$ is a Hilbert-space (see [48], p. 68, where this is proved for the more general case of reflexive Banach-spaces). Using now the sequential continuity we see that K is mapped from (H_1,W) <u>onto</u> sequentially compact, thus compact sets in the metrical spaces $(C^o,||_\infty)$, $(H_o,||_o)$.

Remark: the last considerations outlined a proof that in general a compact (completely-continous) operator defined on a reflexive Banach-space going into a normed vector-space maps indeed closed balls <u>onto</u> compact sets.

Those considerations were made for the sake of completeness and to give a more structural background. What we need essentially is the following theorem which is now an immediate consequence of (iii) (see also [3], p.196, where a more general case is proved)

<u>Theorem:</u> If v is a function in H_o and if there is a sequence v_n of functions in H_1 such that $|v_n - v|_o \rightarrow 0$, $|v_n|_1 \leq C$, then v is in H_1 and $|v|_1 \leq C$.

Calculus of Difference Quotients and Smoothness Criteria

Further we present here several elementary facts from the calculus of difference-quotients, see for instance [17], p.63-64. We follow here closely [15], 127-131. From now on we will consider a function $c:[\bar{a},\bar{b}] \to E^n$, $\bar{a} < \bar{b}$, which has a compact support in $[\bar{a},\bar{b}]$, i.e. $c \equiv 0$ on $[\bar{a},\bar{a}+\bar{h}_o]$, $[\bar{b}-\bar{h}_o,\bar{b}]$, $0 < \bar{h}_o < \frac{1}{4}(a-b)$.
For our further considerations we will work with a restriction of c, i.e. with $c \;:[a,b] \to E^n$, where $a = \bar{a}+h_o$, $b = \bar{b}-h_o$, $h_o = \frac{1}{2}\bar{h}_o$. Those clumsy explanations are necessary if one wants to be precise when working with the shifted vector, which we define by $c_{(h)} : [a,b] \to E^n$, $c_{(h)}(t) := c(t+h)$, $h \leq h_o$, and the difference-quotient $\Delta_h c : [a,b] \to E^n$, $(\Delta_h c)(t) := \frac{c(t+h)-c(t)}{h} = \frac{c_{(h)}(t)-c(t)}{h}$, here we always use the same notation for c and its restriction.
We supply two assertions concerning difference-quotients:

(V1) For $c \in H_1([\bar{a},\bar{b}],E^n)$ we have $|\Delta_h c|_0 \leq |c|_1$ and $|\Delta_{-h} c| \leq |c|_1$, $h < h_o$ and the integrals in the norms taken over $[a,b]$.

(V2) If $c \in H_1([\bar{a},\bar{b}],E^n)$, then $\lim_{h\to 0} |Dc - \Delta_h c|_0 = 0$, where the integral belonging to the norm $|\;|_0$ is again taken over $[a,b]$.

(Note in (V2) we don't need the above general support assumption but $c \in H_1([a,b],E^n)$ is sufficient for (V2).)

We now first prove two assertions which will be helpful when we derive rules for the calculus of difference-quotients.

(A1) Let f be a real-valued function, Lebesgue-summable on $[a-h_o,b+h_o]$, with $f \equiv 0$ on $[a-h_o,a+h_o]$ and $[b-h_o,b+h_o]$, then we have
$$\int_a^b f(t+h) = \int_a^b f(t) \quad \text{if } |h| < h_o .$$

Proof of (A1): By a basic theorem of integration theory (see [24], p.280) we have an absolutely continous function $F : [a-h_o, b+h_o] \to R$ such that $F'(t) = f(t)$ holds almost everywhere on $[a-h_o, b+h_o]$ and $F(x)-F(y) = {}_y\int^x f(t)dt$ for all $x,y \in [a-h_o, b+h_o]$. If we define $\varphi_h(t) := t+h$, $h>0$, then $F \circ \varphi_h$ is absolutely continous on $[a,b]$, (φ_h is Lipschitz-continous), hence $\frac{d}{dt}(F\circ\varphi_h)(t) = f(t+h)$ almost everywhere on $[a,b]$, and

$$\begin{aligned} {}_a\int^b f(t+h)dt &= (F\circ\varphi_h)(b)-(F\circ\varphi_h)(a) \\ &= (F\circ\varphi_h)(b)-F(\varphi_h(b-h))+F(\varphi_h(b-h))-F(\varphi_h(a-h)) \\ &\qquad +F(\varphi_h(a-h))-(F\circ\varphi_h)(a) \\ &= {}_{b-h}\int^b f(t+h)dt + F(b)-F(a) - {}_{a-h}\int^a f(t+h)dt \\ &= F(b)-F(a) = {}_a\int^b f(t)dt . \end{aligned}$$

(Analogous for $-|h_o| < h < 0$)

(A2) Let be $g \in H_o([a-h_o, b+h_o], R)$, then

$$\lim_{h\to 0} \left({}_a\int^b |g(t)-g(t+h)|^2 \right) = 0 , \quad |h| < h_o .$$

Proof of (A2): For arbitraryly given $\varepsilon>0$ there exists a continous function $g_\varepsilon : [a-h_o, b+h_o] \to R$ such that $|g-g_\varepsilon|_o < \frac{1}{3}\varepsilon$. Using the uniform continuity of g_ε, we chose now $h(\varepsilon)$ so small that

$$|g_\varepsilon(t) - g_\varepsilon(t+h)| < \frac{1}{3}\varepsilon|a-b|^{-\frac{1}{2}} \quad \text{for all } |h| < h(\varepsilon), t\in[a,b] .$$

This gives $\left({}_a\int^b |g(t)-g(t+h)|^2 \right)^{\frac{1}{2}} \leq$

$$\leq \left({}_a\int^b|g(t)-g_\varepsilon(t)|^2\right)^{\frac{1}{2}} + \left({}_a\int^b|g_\varepsilon(t)-g_\varepsilon(t+h)|^2\right)^{\frac{1}{2}} + \left({}_a\int^b|g(t+h)-g_\varepsilon(t+h)|^2\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon , \quad \text{thus}$$

$${}_a\int^b|g(t)-g(t+h)|^2 < \varepsilon \quad \text{for all } |h| < h(\varepsilon) .$$

<u>Proof of (V1)</u>: $\Delta_h c(t) = \dfrac{c(t+h)-c(t)}{h} = \dfrac{1}{h} \int_0^h c'(t+s)ds$;

thus we have

$$|\Delta_h c(t)|^2 \leq \left|\frac{1}{h} \int_0^h c'(t+s)ds\right|^2 \leq \frac{1}{h^2} \left(\int_0^h |c'(t+s)| ds\right)^2$$

$$\leq \frac{1}{h^2}\, h \int_0^h |c'(t+s)|^2 ds \quad .$$ This gives

$$\int_a^b |\Delta_h c(t)|^2 \leq \frac{1}{h} \int_a^b \left(\int_0^h |c'(t+s)|^2 ds\right)dt$$

$$= \frac{1}{h} \int_0^h \left(\int_a^b |c'(t+s)|^2 dt\right)ds$$; here the last

equality holds by viewing at $|c'(t+s)|^2 =: f(s,t)$ as a function in two variables over the rectangle $Q = [0,h] \times [a,b]$, where a theorem of Tonelli (see [16], p.51) assures that $f(s,t)$ is a summable function on Q , since for instance $t \to f(s,.)$ is summable on $[a,b]$ for almost all (here even all) s in $[0,h]$ and $F(s) := \int_a^b |f(s,t)|dt$ is summable on $[0,h]$. (Note that by (A1) we get $F(s) = F(0)$!) Then Fubini's theorem (see [16], p.50) gives the last equality above. Further $F(s) = F(0)$ gives

$$\frac{1}{h} \int_0^h \left(\int_a^b |c'(t+s)|^2 dt\right)ds = \frac{1}{h} \int_0^h \left(\int_a^b |c'(t)|^2 dt\right)ds$$

$$= \int_a^b |c'(t)|^2 dt \leq |c|_1^2 \quad .$$

This proves $|\Delta_h c|_0 < |c|_1$, the proof of $|\Delta_{-h} c|_0 \leq |c|_1$ is analogue.

<u>Proof of (V2)</u> : One easily gets

$$Dc(t)-\Delta_h c(t) = \frac{1}{h} \int_0^h \{ c'(t)+c'(t+s) \} ds \quad ,$$ thus

applying the Schwarz-inequality in a similar fashion as in the proof of (V1) leads to

$$|Dc(t)-\Delta_h c(t)|^2 \leq \frac{1}{h^2}\, h \int_0^h |c'(t)-c'(t+s)|^2 ds \quad .$$ Hence

$$| Dc - \Delta_h c |_o^2 = {}_a\int^b | Dc(t) - \Delta_h c(t) |^2 dt$$
$$\leq \frac{1}{h} {}_a\int^b ({}_0\int^h | c'(t) - c'(t+s) |^2 ds) dt$$
$$= \frac{1}{h} {}_0\int^h ({}_a\int^b | c'(t) - c'(t+s) |^2 dt) ds ,$$

where the last equality is proved again by using Tonelli's and Fubini's theorem in a similar way as above. Now

$$\frac{1}{h} {}_0\int^h ({}_a\int^b | c'(t)-c'(t+s) |^2 dt) ds \leq$$
$$\leq \sup_{0\leq s\leq h} {}_a\int^b | c'(t)-c'(t+s) |^2 dt = s(h) ,$$

where $s(h) \to 0$ as $h \to 0$, by (A2).

The preceding considerations in the appendix enable us to present now a theorem which is the central analytic tool for the proof of the smoothness properties of the rectifiable shortest curves, i.e.

<u>Theorem:</u> (Note we will often use the same notation for a function and its restriction!) Let be $c \in H_1([a,b],E^n)$, $a < b$,

A) Let be $a_o < b_o$, $[a_o,b_o] \subset (a,b)$, if there is some constant number M such that ${}_o|\Delta_h c|_1 \leq M$ for all $0 < h < \min\{ |b-b_o| , |a-a_o| \}$, where ${}_o|\ |_1$ stands for the norm $|\ |_1$ with the integral taken over the interval $[a_o,b_o]$, then the derivative c' is contained in $H_1([a_o,b_o],E^n)$. Further we have ${}_o|c'|_1 \leq M$, this means ${}_{a_o}\int^{b_o} |c''|^2 \leq M^2$ and thus ${}_{a_o}\int^{b_o} |c''| \leq M |a_o-b_o|^{\frac{1}{2}}$.

B) Let be $I_i := [a_i,b_i]$, $i \in N$, $a < a_i < b_i < b$ a monotone increasing sequence of intervals exhausting (a,b), $I_i \subset I_{i+1}$ for all $i \in N$ and $(a,b) \subset \bigcup_{i\in N} I_i$. If we have some fixed number M such that for every given I_i ${}_i|\Delta_h c|_1 < M$ for all $0 < h < \min\{|b_i - b|, |a_i - a|\}$ (where the integral in the norm is taken over the interval I_i), then c' is contained in $H_1([a,b],E^n)$ and $|c'|_1 \leq M$.

<u>Proof:</u> A) The statement of A) is an immediate consequence if we combine V_2 on p. 56 with the theorem on p. 55 . The estimations for the integrals ${}_{a_0}\int^{b_0}|c''|^2$ and ${}_{a_0}\int^{b_0}|c''|$ directly follow from the definition of ${}_0|\ |_1$ and by applying the Schwarz inequality.

B) Using A) we can define a sequence of summable functions f_i on $[a,b]$ by $f_i := |c''|^2$ on I_i and $f_i := 0$ on $[a,b] \setminus I_i$ for all $i \in N$. Since ${}_a\int^b f_i < M$ and because $\lim_{i\to\infty} f_i(t) = |c''(t)|$ for almost all $t \in [a,b]$, we get by Fatou's lemma (see [18], p.48) that $|c''|^2$ is summable on $[a,b]$ and ${}_a\int^b |c''|^2 \leq M$. We know by A) that c'(t) exists on all (a,b) and that we have for instance for all $t \in [a_1,b)$

$c'(t) = g(t) = {}_{a_1}\int^t c''(s)\,ds + c'(a_1)$, thus c'(t) agrees almost everywhere on $[a_1,b]$ with an absolutely continous function, for g(t) is absolutely continous on all $[a_1,b]$, being integral of a summable function (see [24],p.280) .

Now c being absolutely continous on all [a,b] can be written as an integral of its derivative: $c(x) = {}_{a_1}\int^x c'(t) + c(a_1) = {}_{a_1}\int^x g(t)\,dt + c(a_1)$ for all $x \in [a_1,b]$. This means c(x) is an integral of an absolutely continous function on all $[a_1,b]$ and by a similar conclusion on all [a,b]. This proves B).

Further we supply some rules for the calculus of difference-quotients which we don't prove since they are got by straightforward calculation. Only for R_1 below one might use A_1 on p. 56.

R_1) Let be $z, w \in H_0([a,b], E^n)$ and let have at least one of the functions z, w compact carrier in (a,b), then we have for all sufficiently small $h > 0$

$$\langle z\, ,\, \Delta_h w \rangle_0 = - \langle \Delta_{-h} z\, ,\, w \rangle_0 \ .$$

R_2) Let be $\bar{a} < a < b < \bar{b}$,

i) If $z \in H_1([\bar{a}, \bar{b}], E^n)$ then for all sufficiently small $h > 0$ the shifted function $z_h \in H_1([a,b], E^n)$,

ii) $(Dz)_h = D z_h$, where D denotes differentiation operator,

iii) D and Δ_h commute, i.e. $\Delta_h(Dz) = D \Delta_h z$,

iv) (product rule) If z, w are (simply) real functions with values in E^n, and f a real function with values in $\mathbb{R}$, then

$$\Delta_h \langle z, w \rangle = \langle \Delta_h z, w_h \rangle + \langle z, \Delta_h w \rangle = \langle \Delta_h z, w \rangle + \langle z_h, \Delta_h w \rangle \quad \text{and}$$

$$\Delta_h \langle fz, w \rangle = \langle \Delta_h fz, w \rangle + \langle f_h \Delta_h z, w \rangle + \langle f_h z_h, \Delta_h w \rangle \ .$$

Appendix B:

Arclength of C^1-smooth curves in a Banach space

For our applications of variational techniques in order to prove the smoothness properties of rectifiable shortest, it was important to know that the two definitions of arclength agreed for absolutely continous curves in a Riemannian manifold, see lemma p. 33 . We will prove here the analogue result for C^1-smooth curves in a normed vector space, our proof seems to be simpler than those found in the literature (moreover the curves may be contained in an infinite-dimensional space). For the proof we need the following assertion also used in the proof of the lemma p. 33 :

Assertion: Let be V,W arbitrary Banach spaces and let be $f:V \to W$ any C^1-smooth mapping. Then we can describe f using the remainder-term of the linear approximation, i.e.

$f(x+h) - f(x) = f'(x)(h) + R(x,h)|h|$, i.e. we define

for $h \neq 0$ $R(x,h) := \dfrac{f(x+h)-f(x)-f'(x)(h)}{|h|}$ and

for $h = 0$ $R(x,0) := 0$.

Here $f'(.):V \to L(V,W)$ is a continous mapping with values in the space of continous linear mappings from V into W , the norm in $L(V,W)$ is defined by $|A| := \sup\{|A(x)| \mid x \in V \ , \ |x| = 1\}$ for $A \in L(V,W)$. We claim now that $R(.,.):V \times V \to W$ is a continous mapping.

Proof: For points with $h \neq 0$ the continuity of R is clear, thus we must check that $R(x,h)$ converges to 0 if (x,h) converges against some point $(x_0,0)$.

$$|R(x,h)| = \frac{|f(x+h)-f(x)-f'(x)(h)|}{|h|} =$$
$$= \frac{|f(x+h)-f'(x)h - f(x+0)-f'(x)0|}{|h|}$$
$$\leq \sup_{0\leq s\leq 1} |f'(x+sh)-f'(x)|$$, where the inequality

is got by the 'Schrankensatz' (see [11],p.58) (generalized mean value theorem). Now because f' is continous, the right side of the above inequality converges to zero if (x,h) converges against $(x_0,0)$.

Theorem: Let $c:I=[0,1]\to V$ be a C^1-smooth curve in a normed vector space V. Then $L(c) = {}_0\int^1 |c'(t)|dt$, where we define $L(c) := \sup\{ L(c,Z) = \sum_i |c(t_i)-c(t_{i+1})| \mid t_i \in Z \ , Z \in \mathcal{P}(I)\}$ with Z being a finite partition of the interval I and $\mathcal{P}(I)$ being the set of all finite partitions of I.

Proof: As in the preceding assertion we may describe for $(s,h) \in I\times I$ $\quad c(s+h) = c(s) + c'(s)h + R(s,h)h \quad (*)$,
thus again $R(s,h) := \frac{c(s+h)-c(s)}{h} - c'(s)$, where we define $c(s+h)$ for $s+h \notin [0,1]$ by the right hand side in (*). Now R(s,h) is continous by the preceding assertion and uniform continous since I x I is compact. We have

$$|{}_0\int^1 |c''(t)|dt - L(c,Z)| = |\sum_i ({}_{t_i}\int^{t_{i+1}} |c'(t)| - |c(t_{i+1})-c(t_i)|)$$
$$= |\sum_i ({}_{t_i}\int^{t_{i+1}} |c'(\alpha_i)| - |c'(t_i)(t_{i+1}-t_i) + R(t_i,t_{i+1}-t_i)\cdot(t_{i+1}-t_i)||| = \oplus\oplus$$

(with $\alpha_i \in [t_i,t_{i+1}]$, using the mean value theorem of the calculus of integration.)

$$\oplus\oplus \leq \sum_i ||c'(\alpha_i)| - |c'(t_i)(t_{i+1}-t_i)|||t_{i+1}-t_i| = \circ\circ \ .$$

Due to the uniform continuity of $c'(.)$, $R(.,.)$, we have for

any $\frac{1}{2}\varepsilon > 0$ a $\delta(\varepsilon)$ such that $|c'(s)-c'(\bar{s})| < \frac{1}{2}\varepsilon$ for $|s-\bar{s}| < \delta(\varepsilon)$ and $|R(s,h)-R(\tilde{s},\tilde{h})| < \frac{1}{2}\varepsilon$ for $|(s,h)-(\tilde{s},\tilde{h})| < \delta$.
Therefore if we choose now any partition $Z = (..\ t_i\ ..)$ with $|t_{i+1}-t_i| < \delta$ for all $t_i \in Z$, we get

$$\circ\circ = \sum_i |\ (|c'(t_i)| + \gamma_i) + (|c'(t_i)| + \beta_i)|\ |t_{i+1}-t_i|$$

with $|\gamma_i|$, $|\beta_i| < \frac{1}{2}\varepsilon$.

This gives $\circ\circ = \sum_i |\varepsilon|\ |t_{i+1}-t_i| = \varepsilon$ and proves the theorem.

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