Delaunay and Regular Triangulations in Space

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Abstract

Lecture by Herbert Edelsbrunner, transcribed by Pedro Ramos and Saugata Basu. The regular triangulation has been popularized by Herbert as the appropriate generalization of the Delaunay triangulation to collections of disks.

This talk was mostly devoted to definitions of Voronoi cells, nerves, simplicial complexes, Delaunay triangulations, circular inversions, stereographic projections, and their applications to computational geometry.

The following open problem was posed at the beginning of the talk.

Suppose there are $2n$ unit disks, $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n$, with centers at $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$, such that, $[|a_i - a_j| \geq |b_i - b_j| \ \ \forall i, j]$, then, does it follow that, $AREA(\cup_i A_i) \geq AREA(\cup_i B_i)$?

This problem was first posed by Poulsen (1954) and Kneser (1955).

An approach that one might take to prove this is to use inclusion-exclusion. Indeed, if the disks come closer together the area of their pairwise intersections increase, and since these get subtracted, the inequality works in the right direction. However, we then have the case of intersections of every triples and now the inequality is in the opposite direction to what we desire in the proof.

Another approach to this problem is to look at the decomposition of the union of the disks, by the Voronoi diagram induced by the centers of the disks.

First, we need some definitions.

Definition 0.1 (Nerve of sets) *Given a collection of sets $\mathcal{A}$, the* nerve *of $\mathcal{A}$ (denoted by $N(\mathcal{A})$) is defined to be $N(\mathcal{A}) = \{\mathcal{X} \subseteq \mathcal{A} | \cap_{\mathcal{A}_\rangle \in \mathcal{X}} \mathcal{A}_\rangle \neq \emptyset\}$.*

Definition 0.2 (Abstract simplicial complex) *A collection $\mathcal{K}$ of sets is an* abstract simplicial complex *if $\sigma \in \mathcal{K}$ and $\tau \subset \sigma$ implies $\tau \in \mathcal{K}$.*

Given a set of points in the plane, we can define the collection of sets $\mathcal{A}$ to be the collection of the Voronoi cells. The nerve of $\mathcal{A}$, then consists of singletons, pairs, and triples of these cells. If we join pairs of points, whose corresponding pair of Voronoi cells occur in the nerve, by line segments we get the dual of the Voronoi diagram, also called the *Delaunay triangulation.* The triangles in the Delaunay triangulation, corresponds to the triples of Voronoi cells occurring in the nerve. Thus the Delaunay triangulation corresponds to a

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geometric realization of the nerve. The geometric realization in this case is the collection of various dimensional simplices whose vertices occur as sets in the nerve.

We can also give another definition of the Delaunay triangulation which depend only on the *local properties* of the points. A simplex is in the Delaunay triangulation *if and only if* the interior of the d-sphere passing through the $(d+1)$ vertices of the simplex does not contain any of the other points.

Definition 0.3 (Simplicial complex) *A collection $\mathcal{K}$ of simplices is a* simplicial complex *if*

1. *$\sigma \in \mathcal{K}$ and τ is a face of σ imply that $\tau \in \mathcal{K}$, and,*
2. *$\sigma_1, \sigma_2 \in \mathcal{K}$, implies that $\sigma_1 \cap \sigma_2 = \emptyset$, or $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.*

Definition 0.4 (Convex hull) *A k-simplex in $I\!R^{\,d}$, $0 \leq k \leq d$, is the* convex hull *of $k+1$ affinely independent points in $I\!R^d$.*

Note that, any subset of $l+1 \leq k+1$ points of a simplex define an l-simplex which is called an l-face of the k-simplex.

We can observe that in the plane, any triangulation of a polygon has the same number of triangles. The situation is not the same in other dimensions. As an example, consider the octahedron in dimension three: if we have a regular octahedron, it can be decomposed in four tetrahedra, but if we twist the vertices in such a way that two of the diagonals do not cut each other, then we obtain five tetrahedra.

Question: what happens if we twist the tetrahedron in such a way that the three diagonals do not cross each other?

We next give the definition of an important transformation.

Definition 0.5 (Inversion map) *The* inversion *with center z is a map, $[^o\colon R^d - \{z\} \to R^d - \{z\}][x \to x^o = \frac{x}{|x|^2}]$*

The following properties of the inversion map are easy to prove.

1. *$(x^o)^o = x$*
2. *If S is a sphere such that $z \notin S$ then S^o is a* sphere*.*
3. *If $z \in S$ then S^o is a* hyperplane*.*

Definition 0.6 (Stereographic projection) *Let π be a d-sphere and let $z \in \pi$. The* stereographic projection *is the map from π to π^o that sends the point $x \in \pi$ to x^o.*

Claim 0.1 (Sphere inversion) *If S is a $(d-1)$-sphere in π then S^o is a $(d-1)$-sphere in π^o (or a $(d-1)$-plane if $z \in S$).*

Proof: Sphere inversion ♡

Let S_1 the sphere passing through S^o and z and let S_1^o its inversion. Then S_1^o is a plane which cuts π in S. This proves the claim. ♠

Let S be a set of points in $I\!R^d$ and let S^o be its image via inversion. If P is the convex hull of the set S^o and $\mathcal{K}$ is the stereographic projection of the boundary complex of P then it can be proved that $\mathcal{K}$ is the Delaunay triangulation of the set S.